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THIS IS VOLUME 1 OF A THREE-VOLUME EXPERIMENTAL EDITION CONTAINING A SEQUENCE OF ENRICHED MATERIALS FOR SEVENTH-GRADE MATHEMATICS. THESE MATERIALS ARE DESIGNED FOR A PROGRAM OF INDIVIDUALIZED INSTRUCTION FOR THE ACCELERATED STUDENT OR FOR CLASSROOM PRESENTATION BY THE TEACHER. THE PRESENTATION OF THE MATERIAL IS IN SUCH A MANNER AS TO REFLECT CHANGES IN CONTENT, TECHNIQUE, APPROACH AND EMPHASIS. INSTRUCTIONAL UNITS ON A NUMBER OF SEQUENTIALLY RELATED TOPICS ARE STRUCTURED TO INCORPORATE MODERN TERMINOLOGY WITH THE TRADITIONAL TOPICS AND TO INTRODUCE NEW CONCEPTS AS APPROPRIATE. THIS VOLUME INCLUDES MATERIALS FOR (1) PLANNING A MATHEMATICAL PROCESS, (2) FINITE NUMBER SYSTEMS, (3) SETS AND OPERATIONS, (4) MATHEMATICAL MAPPINGS, (5) INTEGERS, AND (6) PROBABILITY AND STATISTICS. (RP)

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SEVENTH YEAR MATHEMATICS

(Experimental Edition)

Volume 1

Secondary School Mathematics Curriculum Improvement Study

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
OFFICE OF EDUCATION

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Volume I

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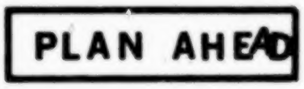
CHAPTER 0: Planning a Mathematical Process

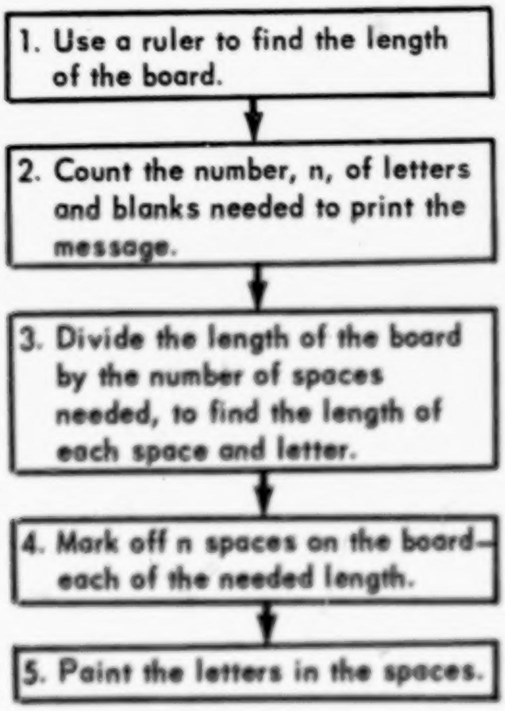
0.1 Flow Charts

Many of you have seen the humorous sign



The humor, of course, is in the fact that the painter of the sign clearly did not heed the advice which he was giving to others. What can the sign painter do to avoid the embarrasment of crowding the letters "E", "A", and "D" on future signs? Either he must develop a better method of sign painting or get instructions from a more experienced painter.

What if we were asked to write detailed instructions to such a novice sign painter for painting the words "PLAN AHEAD" on a piece of cardboard of a given size? We would want the painter to complete the sign without trimming the cardboard or squeezing letters as he did in the sample sign above. We also want space before the first word, between words, and after the last word. Therefore, our instructions probably would be something like the following:



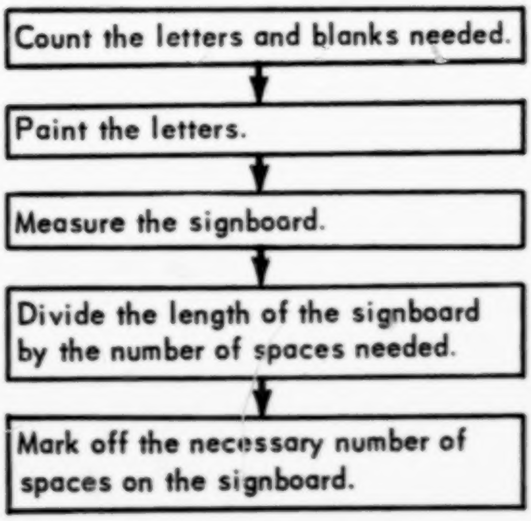
1. Use a ruler to find the length of the board.
2. Count the number, n, of letters and blanks needed to print the message.
3. Divide the length of the board by the number of spaces needed, to find the length of each space and letter.
4. Mark off n spaces on the board–each of the needed length.
5. Paint the letters in the spaces.

Figure 0.1

What will the number of letters and blanks be for "PLAN AHEAD"? If the length of the sign is to be 12 inches, how long will each letter and space be? Are the instructions clear and complete? If not, what modifications should be made?

To make the set of instructions usable for the painter, we might write them out as before on a single sheet of paper, or we could write each instruction on a separate numbered card. A set of instructions given in this form–where individual instructions are recorded on cards or in boxes arranged according to some plan–is called a flow chart. Two things are important in any flow chart–the instructions themselves and their arrangement plan. If either the instructions or their arrangement is altered, the directions may be radically changed.

For instance, suppose instruction 2 was changed to read, "Count only the letters in 'PLAN AHEAD' ". What problem would this cause for the sign painter? If the arrangement plan is changed, problems may arise also. What sort of sign might be produced if the flow chart below was followed?



Count the letters and blanks needed.

Paint the letters.

Measure the signboard.

Divide the length of the signboard by the number of spaces needed.

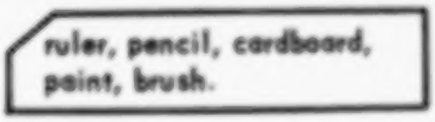
Mark off the necessary number of spaces on the signboard.

Figure 0.2

(Notice that the arrangement plan in this flow chart is indicated by the direction of the arrows)

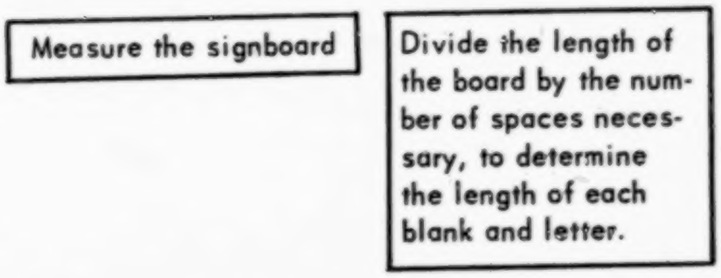
Flow charts are useful whenever a sequence of instructions or commands is necessary. They are particularly useful in mathematics, especially in work with electronic computers. Although computers can perform complex mathematical operations at high speeds, the computer must first receive a detailed sequence of instructions. Since flow charts are useful guides in writing such instructions, mathematicians who prepare programs for these computing machines have developed a standard format and language for the construction of flow charts.

Facts and equipment important for specific processes are recorded in data boxes like this:

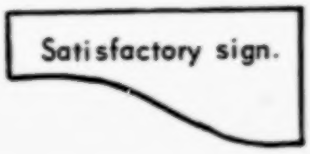


These data boxes or cards are used to record the process input—the information or equipment necessary to carry through the process. Is the card above an appropriate input for the sign painter's flow chart?

Instructions to be carried out or operations to be performed are recorded in operation boxes like these:


Measure the signboard
Divide the length of the board by the number of spaces necessary, to determine the length of each blank and letter.

Information obtained by means of the process or processes described in the flow chart is recorded in output boxes like this:


Satisfactory sign.

(Note that each type of box is given a characteristic shape.)

Using these conventions of notation, try to complete the following partial chart for the sign painting process.

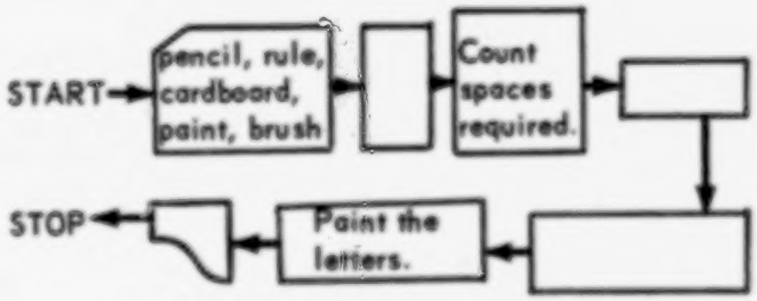

START
pencil, rule, cardboard, paint, brush
Count spaces required.
STOP
Paint the letters.

Figure 0.3

The content and ordering of the instruction cards or boxes are both important. However, it is often possible to give instructions in several ways, each producing the same result. Is there any rearrangement of the instructions in the PLAN AHEAD flow chart that would still produce an acceptable sign? It is possible to expand the instructions within one of the boxes, developing a flow chart within a flow chart. For instance, we could replace the box which commands, "Count the number, n, of letters and blanks needed to print the message," by the small sub-flow chart:

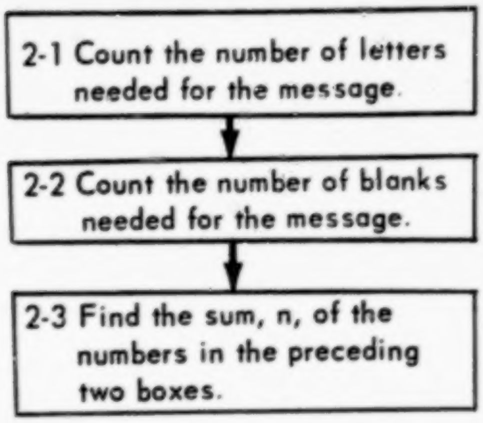

2-1 Count the number of letters needed for the message.
2-2 Count the number of blanks needed for the message.
2-3 Find the sum, n, of the numbers in the preceding two boxes.

Figure 0.4

What other such expansions can you suggest to make the directions clearer?

The flow chart is a useful device for describing complicated processes systematically. It has primary applications in programming for computers, but it is also of use in outlining a wide range of step-by-step procedures. A few of these uses of flow charting are illustrated in the following exercises.

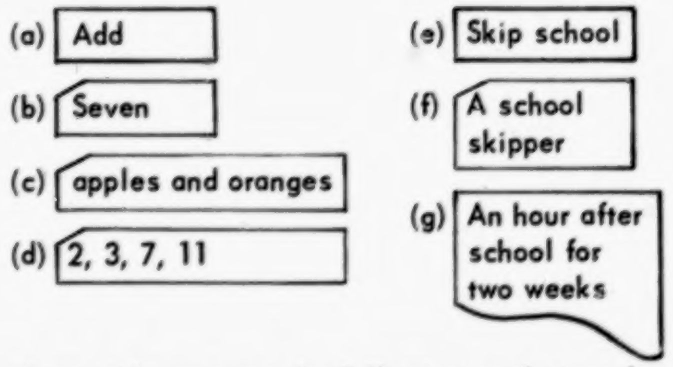
0.2 Exercises

1. Try to construct a flow chart of the main steps describing:
   (a) how you got to school today;
   (b) your daily schedule at school;
   (c) how to draw a circle with compasses or a string and pencil;
   (d) finding the average of two numbers;
   (e) finding the factors of a number;
   (f) a game you play.
   Does the order of the boxes affect the chart you have constructed? If possible, show two orderings that produce the same result.
2. (a) Do the directions in your sign painter's flow chart apply only to the PLAN AHEAD sign?
   (b) Is it possible to use the same chart to produce different signs by use of an additional data box:

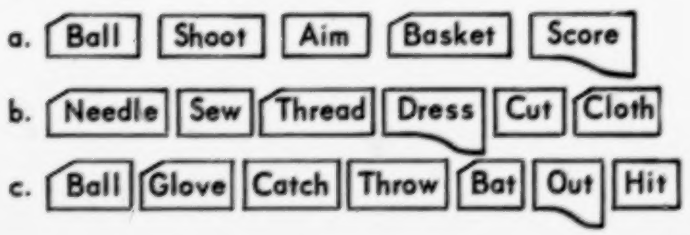
   If so, where should this data box be placed in the chart?
   (c) What would the message boxes be for painting
      (1) CAUTION: RUTABAGAS;
      (2) U.N.C.L.E.;
      (3)__________ __________(your name)?
   (d) What number of letters and blanks is needed in each sign suggested in (c)?

(e) If you have a signboard 24 inches long, how long will each letter and blank be in each of the three signs?

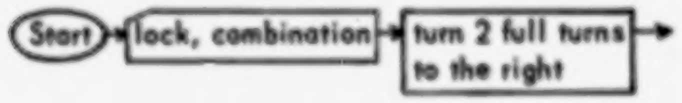
3. Classify the following as input, output, or operation cards:


(a) Add
(b) Seven
(c) apples and oranges
(d) 2, 3, 7, 11
(e) Skip school
(f) A school skipper
(g) An hour after school for two weeks

4. If possible, arrange the following cards in order to give a flow chart that makes sense.


a. Ball
Shoot
Aim
Basket
Score
b. Needle
Sew
Thread
Dress
Cut
Cloth
c. Ball
Glove
Catch
Throw
Bat
Out
Hit

5. Try to write a flow chart for multiplying two fractions which can be followed successfully by someone who doesn't know what a fraction is.
6. Can you write directions for the process of opening a combination lock? You might begin with


Start
lock, combination
turn 2 full turns to the right

0.3 Branching and Looping in Flow Charts

In the preceding section we were introduced to flow charting of simple activities. The flow charts consisted of input, output, and operation boxes which were arranged in a definite order. Often, in more complicated flow charts, you will find a fourth kind of box. This new type of box is used to record things which must be decided before continuing. It is called a decision box and usually looks like this:

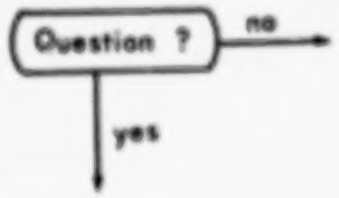

Question ?
no
yes

Figure 0.5

To see the role of the decision box in a flow chart, look at the problem which Jack Anderson faced one morning. When Jack went out to ride his bike to school, he found that the back tire was flat. Fortunately, Jack's father said he would take the bike down to the garage to have the tire fixed. He warned Jack that some of the workers at the garage were not familiar with bicycle tires like those on Jack's special racing bike. He suggested that Jack write out the clearest possible set of directions for what he wanted done.

Jack composed the following flow chart. Read it carefully to see if his flow chart agrees with the one you would have composed. It probably can be improved! After studying it, try to write out a clearer and more complete flow chart.

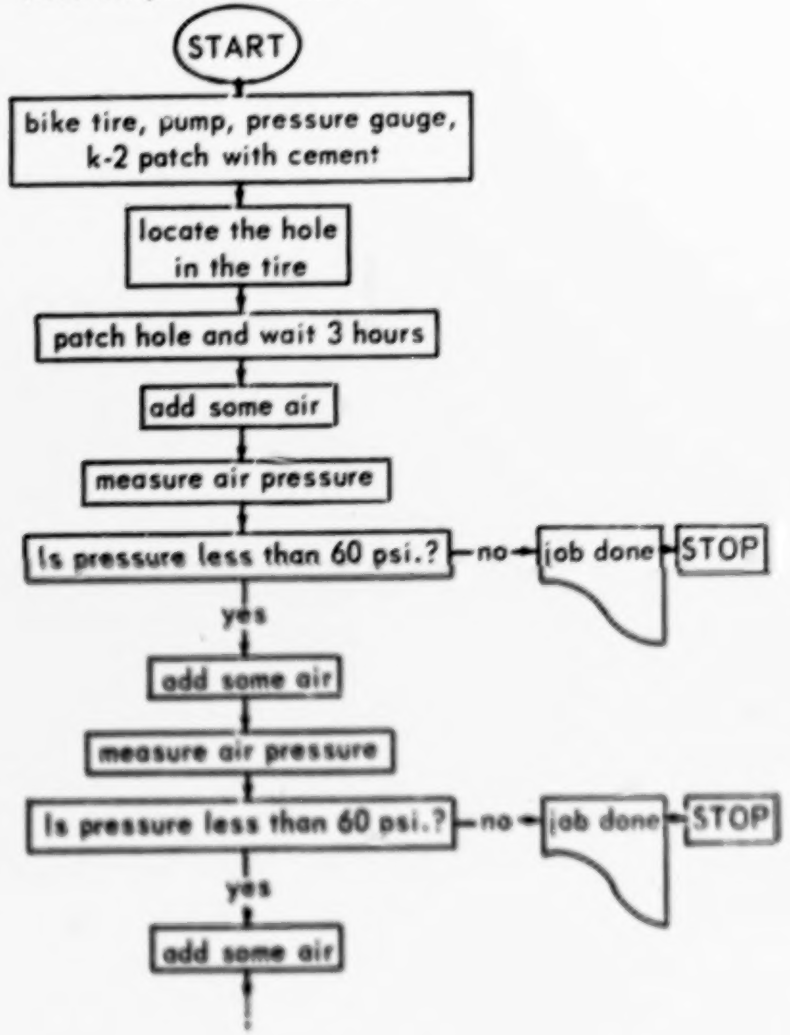

START
bike tire, pump, pressure gauge, k-2 patch with cement
locate the hole in the tire
patch hole and wait 3 hours
add some air
measure air pressure
Is pressure less than 60 psi.?
no
job done
STOP
yes
add some air
measure air pressure
Is pressure less than 60 psi.?
no
job done
STOP
yes
add some air

Figure 0.6

Notice that in each of the decision boxes in the preceding flow chart, a question is asked. The answer to each question—either yes or no—determines the next step. If in the first decision box the answer is "yes", more air must be pumped into the tire. If the answer is "no", the inflation will stop because the correct pressure of 60 psi will have been reached.

Thus, each decision box creates a branch in the flow chart.

Once the hole in the tire has been patched, the flow chart for inflating the tire appears to go on forever in a sequence of "measure, decide, add air, measure, etc." (this indefinite extension of the process is indicated by the three dots "⋮" at the bottom). However, after one cycle has been completed, the instructions simply repeat themselves. At a certain point in the process we go back to an earlier stage and "try again" or repeat part of the instructions. This process of repeating certain steps is called looping. The steps that are repeated are called a loop. Often, instead of rewriting a loop several times (as was done in figure 10.6), we simply connect the parts of the loop with an extra arrow which indicates, "go back and do these steps again."

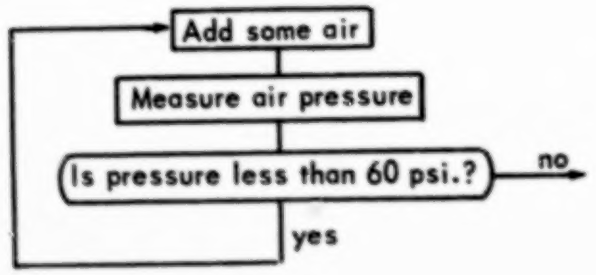

Add some air
Measure air pressure
Is pressure less than 60 psi.?
no
yes

Figure 0.7

Each time a loop is carried out we say that one iteration has been completed.

Anyone who has used a strong pump to inflate a bicycle tire will notice a serious omission in Jack Anderson's flow chart. His decision boxes asked only, "Is the pressure less than 60 psi.?" If the repairman followed the instructions in detail, he might decide to put plenty of air in the tire to make sure it gets up to at least 60 psi. immediately. The results could be disastrous!

What Jack really wants is pressure near 60 psi.—perhaps between 58 psi. and 62 psi.. Therefore, it might be wise to make provision in the flow chart for releasing excess air. What kind of box would this introduce? An input box? A decision box? An operation box? An output box? A combination of these types?

Try to fill in all the missing details of the revised flow chart for fixing Jack's flat tire.

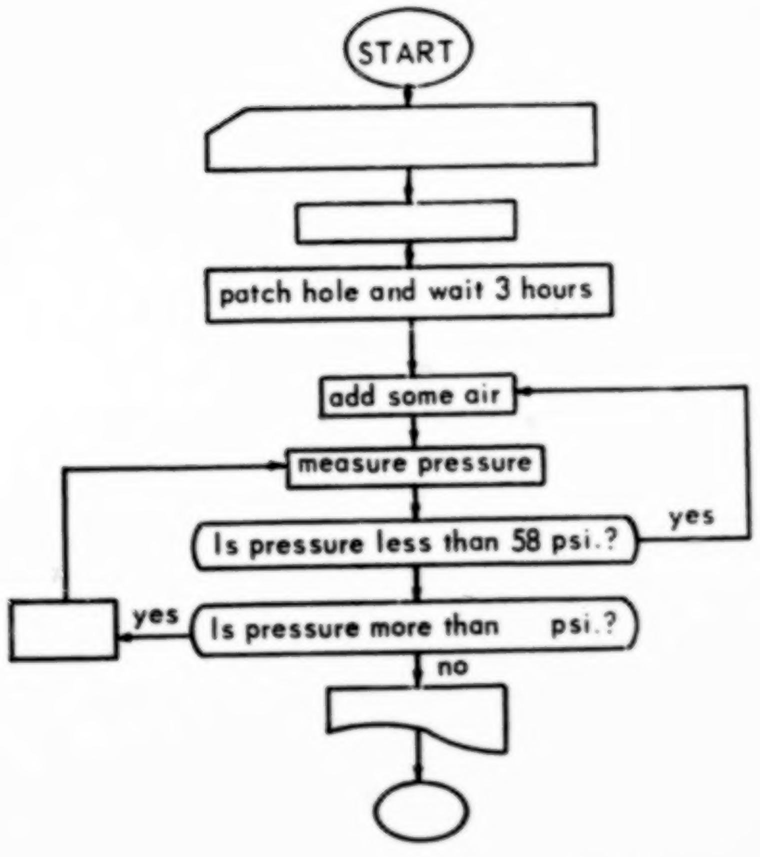

START
patch hole and wait 3 hours
add some air
measure pressure
Is pressure less than 58 psi.?
yes
yes
Is pressure more than psi.?
no

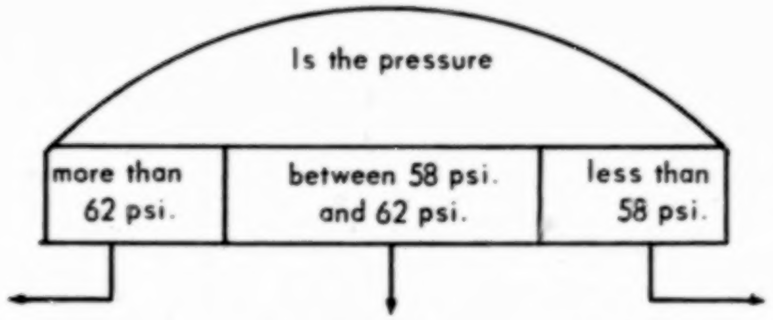
Figure 0.8

In what ways can the arrangement of the flow chart be altered without changing the effect of the instructions? Can you identify at least two loops in the process? What further improvements can you suggest?

The sign painter who wasn't able to follow the advice he was painting on the board and the service station attendant who needed instructions on how to patch a tire each had to have a large job divided into a sequence of smaller tasks. While the problems these men faced may not be typical of those you will face in mathematics or science, the flow charting techniques we have developed to deal with these two problems give us a valuable tool for translating complex problems into sequences of simple tasks.

0.4 Exercises

1. How would Figure 10.8 be changed if the following decision box was to be used instead of the two that are there?


Is the pressure
more than 62 psi.
between 58 psi. and 62 psi.
less than 58 psi.

Can you see any problems in putting such a box in a flow chart?

2. Describe several systematic processes that might involve repetition of certain steps or looping. Write a flow chart which shows the looping for at least one of the processes.

3. Which of the following might properly appear in a decision box?

a. Is it raining?
b. 3 + 2.
c. How are you?
d. I am fine.
e. Supercalifragilistic-expialidotious.

4. Write a branching flow chart for determining the largest (or smallest) number of a set of numbers, {x,y,n}.

5. Write a looping flow chart for finding all divisors of a whole number.

6. Write a looped flow chart giving instructions on how to divide a line in half, quarter, eighth, sixteenth, ...

*7. Can you write a looped flow chart for the process of opening a combination lock when the combination is unknown? You might want to try it first with a lock having only 3 or 4 numbers on the dial.

*8. Write a flow chart describing a defensive strategy in the game of tic-tac-toe.

*These are special challenge problems.

0.5 Flow Charting Addition

In your previous study of mathematics, you spent a good deal of time learning and practicing rules for adding, subtracting, multiplying, and dividing both whole numbers and rational numbers (1/2, 2/3, 5/8, ...). One of the most important features of modern electronic calculators and computers is the fact that these machines can perform the fundamental operations of arithmetic (+, −, ×, ÷) with lightning speed. No longer need man struggle over such computations as

$$\begin{array}{r}87693\\54219\\83752\\+\underline{12374},\end{array}\qquad\begin{array}{r}73652\\\times\underline{4521},\end{array}\qquad\text{or}\quad 6315\overline{)1978247}.$$

See how quickly you can perform these computations. Can you do any one of them in less than 15 seconds? A computer could do all three in less than 1/1000 of one second.

Unfortunately, things aren't quite as simple as they seem at first glance. Despite the fact that computers can perform a variety of simple operations almost instantaneously, like the sign painter or the tire repairman, the computer cannot solve problems until they are broken into a sequence of very simple steps. If we hope to have the computer perform our adding, subtracting, multiplying, and dividing, we must write flow charts or programs telling the machine what steps to take.

Let's begin by trying to flow chart some operations on the set of whole numbers, {0, 1, 2, 3, 4, ...}. You recall that the three dots "..." indicate that the set is endless or infinite. For convenience, we will use the symbol "W" to represent the set of whole numbers. Thus,

$$W = \{0, 1, 2, 3, 4, 5, \ldots\}.$$

Each whole number is said to be an element of W.

In order to write a flow chart describing the process of adding whole numbers, it is necessary to take a close look at the steps involved in performing this computation. How would we instruct an ignorant machine (or person) to add the whole numbers 562 and 395?

Following the usual method we would proceed like this:

$$\begin{array}{r}562\\+\underline{395}\end{array}$$

Step 1) 2 + 5 = 7;
Step 2) 6 + 9 = 15; write down 5 and carry 1;
Step 3) 5 + 3 = 8; 8 + 1 = 9;
Therefore, the answer is 957.

To someone who has worked with similar addition problems, there would be little question that the above directions are correct. But could these instructions be understood by a person who had never seen an addition problem? This person might ask, "Why isn't the answer 5, since that is the only number that was recorded?" Or, "Where did the 1 come from in Step 3)? Why did you add the particular pairs of numbers you chose—why not 5 + 9, 6 + 5, and 2 + 3? What does 'carry 1' mean?"

These same questions and many others would certainly confuse a machine with no prior directions on adding. This machine wouldn't even know that 2 + 5 = 7!

We have found that flow charts often make step-by-step procedures more clear. If you were to try writing the steps of the above computation in the form of a flow chart, your chart would probably look something like this!

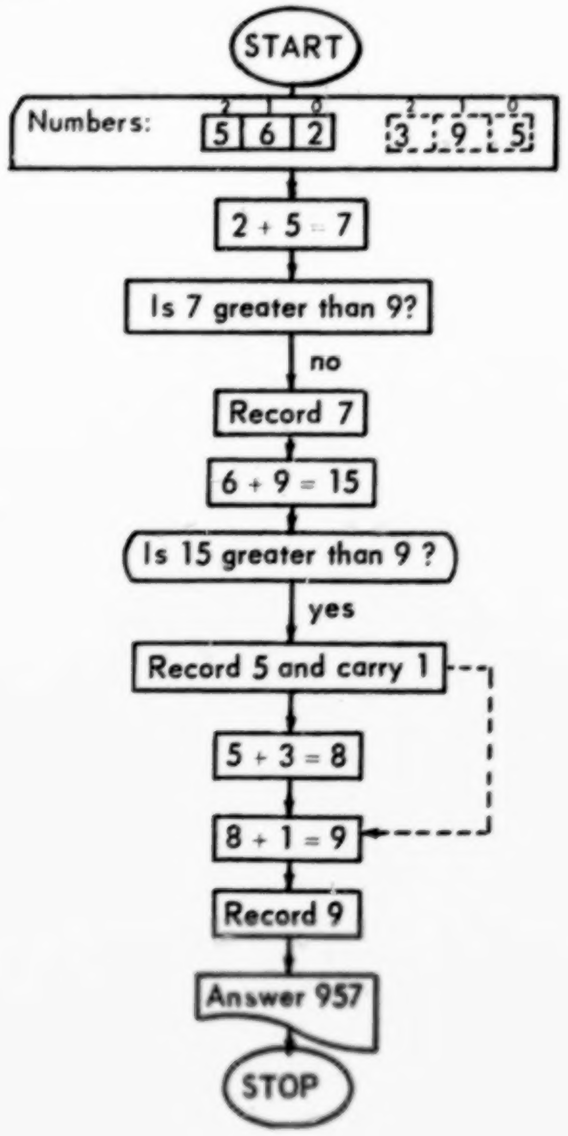

START
Numbers:
2 1 0
5 6 2
2 1 0
3 9 5
2 + 5 = 7
Is 7 greater than 9?
no
Record 7
6 + 9 = 15
Is 15 greater than 9 ?
yes
Record 5 and carry 1
5 + 3 = 8
8 + 1 = 9
Record 9
Answer 957
STOP

Figure 0.9

This flow chart gives a clear picture of the order of steps in the computation of 562 + 295, but is it of any use in directing calculation of other sums?

The flow chart in Figure 0.10 gives general directions for computing the sum of a pair of three digit numbers. The frames in the input box indicate the numbers to be added: "$\overset{2\ 1\ 0}{\square\square\square}$" the first number and "$\overset{2\ 1\ 0}{⬚⬚⬚}$" the second. Thus the entry in "$\overset{0}{\square}$" would be the ones digit of the first number, the entry in "$\overset{1}{\square}$" would be the tens digit of the first number, the entry in "$\overset{2}{⬚}$" would be the hundreds digit of the second number, etc.

Study this flow chart carefully to see whether it gives clear instructions on how to add numbers like 562 and 395.

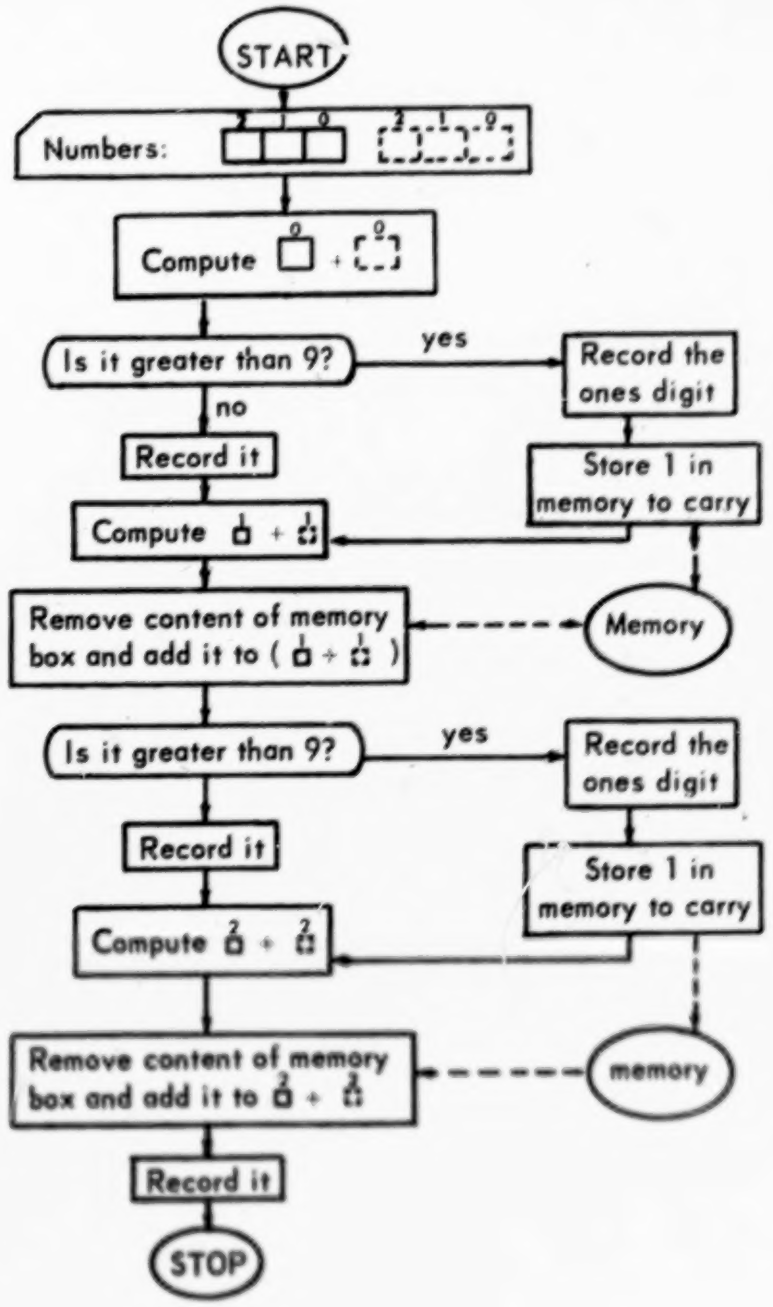

START
Numbers:
2 1 0
2 1 0
Compute 0 + 0
Is it greater than 9?
yes
no
Record the ones digit
Store 1 in memory to carry
Record it
Compute 1 + 1
Remove content of memory box and add it to (1 + 1)
Memory
Is it greater than 9?
yes
Record the ones digit
Store 1 in memory to carry
Record it
Compute 2 + 2
Remove content of memory box and add it to 2 + 2
memory
Record it
STOP

Figure 0.10

If you had trouble understanding the instructions of the chart in figure 0.10, try to complete the example begun below.

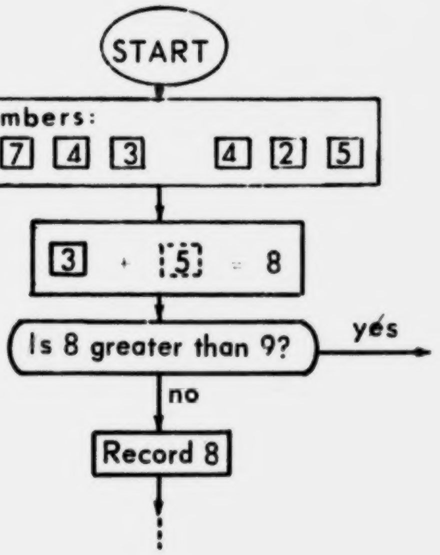


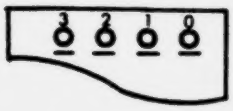
Figure 0.11

Do the results you obtained in working out this example indicate that the flow chart is correct? If not, can you see how to alter it to make the necessary corrections? Any place where the directions are incomplete or unclear should be improved.

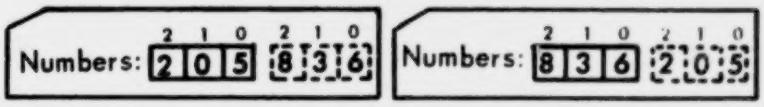
Although it is important to develop a flow chart which produces "right answers", it is equally important to understand why the flow chart produces these answers. Without understanding this theory behind the flow chart, it is impossible to have confidence that the procedures outlined will always give the correct answer. The following exercises raise some of these "why" questions about the procedure for addition or addition algorithm.

0.6 Exercises

1. One shortcoming you may have found in the flow chart is the lack of direactions on how or where to record the digits. How might the following output box be built into the chart to take care of this problem? Must changes be made in some other boxes to make the use of this output box clear?

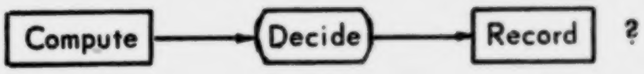


2. Why does the flow chart contain instructions to, "store 1 in memory to carry" and later to "remove content of carry and add it to $\square + [\,]$ "? What does it really mean to carry 1?
3. Explain why the pairs of numbers to be added are chosen in the way they are. That is, why must $\overset{1}{\square}$ be added to $\overset{1}{[\,]}$ and not to $\overset{0}{\square}$ or $\overset{0}{[\,]}$?
4. In every computation box of the flow chart, you are directed to compute $\square + [\,]$. What would happen to the result if we wrote $[\,] + \square$ instead? Why?
5. Can you explain why 8756 + 3241 = 8251 + 3746 but 8756 + 3241 $\neq$ 8152 + 3647? (Notice how certain digits of the first pair have been interchanged to produce the second and third pairs)
6. Follow the flow chart to completion with the input cards



Do the two results agree? If not, why not? Compare your answer here to that of problem 4.

7. How might the addition flow chart be modified to introduce a loop to avoid the repetition of the cycle



How can it be modified to add four digit numbers? Can the flow chart given be used to add 2 digit numbers?

8. If we assume that the user of our addition flow chart knows no addition facts, we must give him a small table and directions on how to use it to compute $\overset{0}{\square} + \overset{0}{[\,]}$, ... How large a table would be necessary? How would you revise the chart to introduce this change?
9. If you are confident that you have a flow chart that breaks long addition into small steps anyone can do, try it out on younger brothers or sisters who haven't seen problems like 395 + 413 before.
10. What change would result if a new operation box $\boxed{\text{Add 0}}$ was introduced between each of the computation and record boxes? Why?

0.7 A Flow Chart for Subtraction

Having successfully developed a flow chart for addition, it is natural to seek similar charts for subtraction, multiplication, and division. From arithmetic you recall the close relationship between addition and subtraction—the two processes can, in a sense, be described as "equal but opposite". Thus, we would suspect that a workable flow chart for subtraction can be obtained from our addition chart by simply interchanging "add" and "subtract", "+" and "–", and "carry" and "borrow".

To see whether this will work, let's look at the subtraction algorithm applied to a typical problem, 954 – 283. One way our computation might proceed is as follows:

1) 4 – 3 = 1; write down 1

2) 8 is greater than 5; borrow 1 from 9 making 9 into 8 and 5 into 15

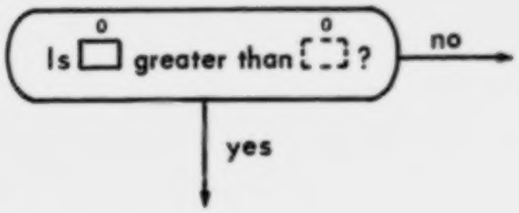
3) 15 – 8 = 7; write down 7

4) 8 – 2 = 6; write down 6

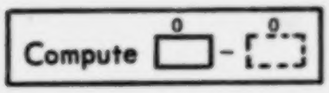
Therefore, the answer is 671.

Although the interchange of terms suggested above does look promising for developing a subtraction flow chart, there are several obvious places where new steps must be introduced and others where directions must be completely rewritten.

The first place a new step is needed is between the input box containing the numbers and the operations box which commands, "compute – ." We need a decision box



Of course, if the answer here is yes, we can proceed to



If the answer is no, we must give instructions on how to perform the borrowing and the subtraction of $\overset{0}{⬚}$ from a revised $\overset{0}{□}$. What form would these borrowing instructions take?

At this point, our flow chart will look something like the following:

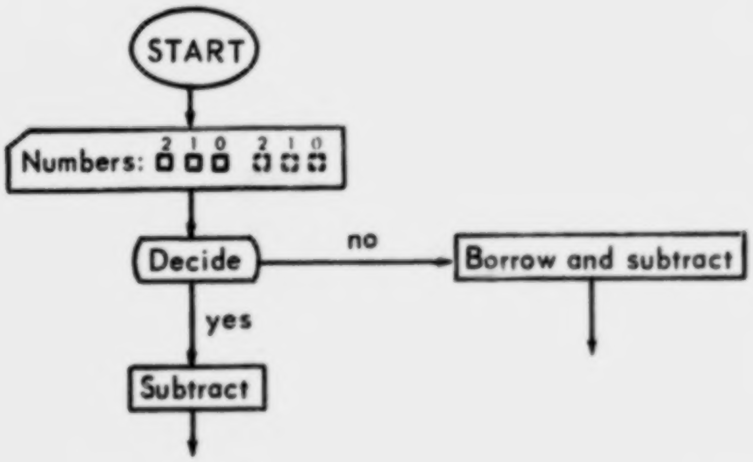

START
Numbers: $\overset{2}{□}\overset{1}{□}\overset{0}{□}$ $\overset{2}{⬚}\overset{1}{⬚}\overset{0}{⬚}$
Decide
no
Borrow and subtract
yes
Subtract

Figure 0.12

The next logical step is to record the result of our first computation. In the recording step of our addition flow chart, we had to first decide if $\overset{0}{□}$ + $\overset{0}{⬚}$ was greater than 9. Is $\overset{0}{□}$ – $\overset{0}{⬚}$ ever greater than 9? If so, how can we write the chart in order to record this number? If not, what is the next step in our subtraction flow chart if we are on the main stem of the chart? If we are on the branch?

You may have noticed that in the addition flow chart of section 0.5 there is a kind of looping. Between successive computation boxes, the steps are the same–add, carry, decide, record–and within these boxes, only the □ and ⬚ change. Thus when one loop has been completed, the rest of the flow chart can be obtained by repeating the pattern of the loop. Will the same repetition occur in a flow chart describing subtraction? For instance, decide, subtract, register, decide, ... ? If not, what modifications should be made to meet the different requirements of the subtraction algorithm? Now, referring to the addition flow chart and the suggestions of this section as guides, try to write out a complete flow chart for subtracting one three digit number from another.

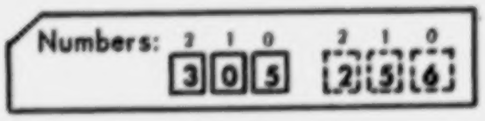
0.8 Exercises

1. Use your chart to compute 456 – 292. Does your chart produce the correct answer? If not, where did it go wrong?
2. Check the following computations to see if they are correct. If not, explain the error and show the correct computations.

a. $\begin{array}{r} 943 \\ -765 \\ \hline 288 \end{array}$ b. $\begin{array}{r} 82 \\ -19 \\ \hline 63 \end{array}$ c. $\begin{array}{r} 1523 \\ -1237 \\ \hline 386 \end{array}$

3. Refer back to the example 954 – 283 in section 0.7:
    a. What does it really mean to say, "Borrow 1 from 9 making 9 into 8 and 5 into 15"?
    b. When we wrote down 1, 7, and 6 in that example, why did they go in the order 671? Why not take 761 or 176?
    c. Why did we try to subtract 8 from 5? Why not take 8 from 9 and 2 from 5 to avoid all the borrowing bother?

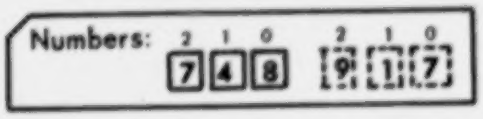
    (Hint: You might want to use the fact that $954 = 9(100) + 5(10) + 4$ and $283 = 2(100) + 8(10) + 3$.)
4. What would the input card be if we wanted to perform 75 – 39? The output card?
5. What modifications are necessary to make your subtraction flow chart work for four digit numbers?
6. Try using your flow chart to compute the difference of the numbers on this input card.



Does your chart allow for borrowing twice? If not, what changes will correct this fault?

0.9 Operations and Non-operations

What would a user of your subtraction flow chart do if given the input card



Following the systematic procedures you have given, he would probably work as follows:

1) 8 – 7 = 1; record 1
2) 4 – 1 = 3; record 3
3) 9 is greater than 7;
borrow 1 from, ... ?
HELP !!!

He has been asked to perform a computation for which his flow chart gives incomplete instructions. Is it possible to give instructions for computing 748 – 917? Is there a whole number which can be added to 917 to give 748? It appears that we may need a new decision box in our flow chart, right at the beginning.

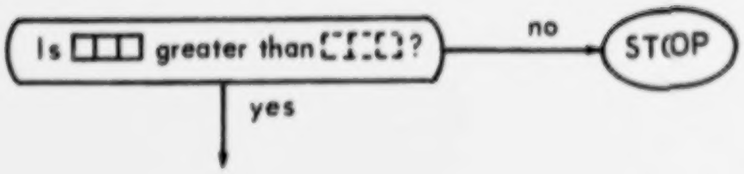


Figure 0.13

Although we said that addition and subtraction are, in a sense, "equal but opposite", the above difficulty illustrates a basic distinction between the two processes. If we select any two whole numbers in a certain order, we can always assign to this ordered pair one and only one whole number called their sum. As we have just seen, there are some ordered pairs of whole numbers whose difference is not a whole number.

The key to this distinction between addition and subtraction is the word ordered. Certainly if we were given the numbers 748 and 917, we could find the difference 917 – 748. But the problem was to find 748 – 917. The order of the two numbers given was significant. (If problems such as 748 – 917 seem unrealistic, consider the situation of a man who has $748 in his checking account, writes a check for $917, and tries to compute his account balance. This happens!)

In order to indicate the distinction between an "always possible" and a "sometimes possible" process, mathematicians have adopted some conventions of terminology. To always possible processes such as addition, they apply the term binary operation. The set of whole numbers with the operation of addition is designated by (W, +), and is called an operational system. Since subtraction of whole numbers is a sometimes possible process, it is not called an operation on the whole numbers.

An important part of our future work in mathematics will be the study of operational systems and their properties. In the exercises of section 0.6, you encountered two important properties of the operational system (W, +)

(1) If a and b are any whole numbers,

$$a + b = b + a$$

(Commutative Property of Addition)

(2) If a is any whole number,

$$a + 0 = 0 + a = a$$

(Property of Zero for Addition)

Because zero has property (2), it is called the identity element for (W, +) or the additive identity for whole numbers. The importance of these two properties of W will become clear later when we meet operations which are noncommutative and operational systems with no identity element.

0.10 Exercises

1. Is multiplication an "always possible" process in the set of whole numbers? If not, why not? If so, what symbol would you suggest for the corresponding operational system?
2. Is division an operation in W? Why or why not?
3. Addition was found to be a commutative operation on W. Is subtraction commutative? Why or why not?
4. Is multiplication a commutative operation on W? Why or why not? What about division?

0.11 Flow Charting Multiplication

A computer which knows how to add can be quickly taught to multiply as well. You know from arithmetic that 2 × 7 = 7 + 7, 4 × 13 = 13 + 13 + 13 + 13, 3 × 164 = 164 + 164 + 164, etc. This nice relationship between addition and multiplication does not hold for all number systems–try writing 2/3 × 1/2 using addition. However, it can be used to multiply whole numbers if addition can be performed very quickly. Before reading farther, try to write a looped flow chart for multiplication using the relationship between addition and multiplication illustrated in the above examples. You might find it convenient to introduce a counter to keep track of the number of iterations or times you have performed the desired addition.

If you succeeded in writing a workable flow chart for multiplication in terms of addition, a computer could use it to compute

$$\begin{array}{r} 7592 \\ \times\ 83 \\ \hline \end{array}$$

by adding: $\underbrace{7592 + 7592 + \ldots + 7592}_{83 \text{ summands}}$. However, this is not a method anyone would want to use if no such high speed machine was available. The usual multiplication algorithm would be a far simpler way to find the indicated product.

So far, we have developed flow charts for the addition and subtraction algorithms by studying sample problems in order to identify the sequence of steps involved in each process. Work out the example 7592 × 83 – writing down each step of the procedure as we have done earlier:

Step 1) 3 × 2 = 6; record 6
Step 2) 3 × 9 = 27; record 7; carry 2

Do you see similarities in the algorithms for multiplication, addition and subtraction? If so, which pair are most similar? Does "carry" mean the same thing here that it meant for addition? What is the pattern to be followed for "recording"?

The first several steps of a multiplication flow chart are given in Figure 14. Study them carefully, copy them on a separate paper, and try to finish the chart. (Assume that any user of the chart will know how to add.)

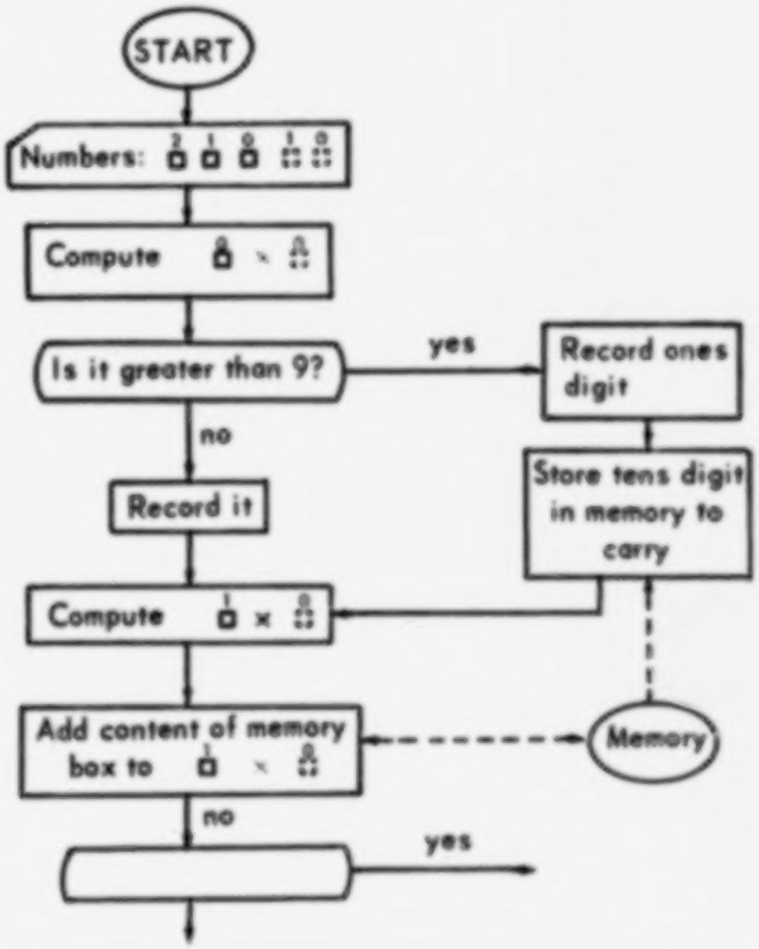
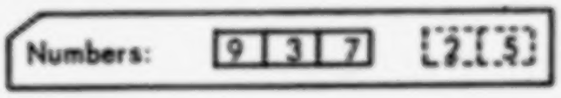


Figure 0.14

If people who didn't know the multiplication algorithm were to use your flow chart, there is one kind of box in the chart that would probably consistently lead to errors and confusion. This is the "Record" box. In your own completed chart you may have revised or expanded the instructions in these boxes, but if you didn't, the directions are insufficient.

Consider the problem with input card



Applying the multiplication algorithm your computation would probably look like this:

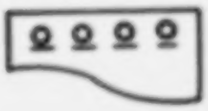
$$\begin{array}{r} 937 \\ \times\ 25 \\ \hline 4685 \\ 1874 \\ \hline 23425 \end{array}$$

Therefore, 937 × 25 = 23425. Are directions in the record boxes clear enough to guarantee that this answer will be obtained? Someone might write down all the right numerals in a nonsensical order.

4 7 5
8
8 1 6 or 58644781
4

How can clear directions for recording be written into your flow chart for multiplication?

You may recall that in sections 0.5 and 0.6 we also met the problem of how to record digits. Exercise 1 of section 0.6 suggests using an output box like



Can you use such an output box or combination of output boxes to write clear instructions for each recorder box?

Even if we develop a workable recording plan, there is one more important problem. The last several steps of your flow chart would probably be

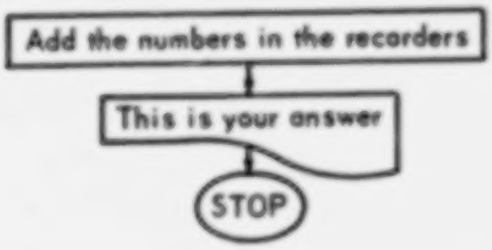


Figure 0.15

Following these directions in the example above, we would compute

$$\begin{array}{r} 4685 \\ +\ 1874 \\ \hline 6559 \end{array}$$

and report the answer as 6559. How would you instruct someone to add 4685 and 1874 in the proper form

$$\begin{array}{r} 4685 \\ +\ \underline{1874} \end{array} \quad ?$$

Or more important, why should the numbers be added with the 1874 shifted one place to the left?

The answers to both of these questions can be found by carrying out the multiplication in an expanded form as follows:

1) Note that 937 × 25 = (937 × 20) + (937 × 5).

2) We can expand each of the terms on the right side of the "=" sign once again:

$$\begin{aligned} 937 \times 20 &= (900 \times 20) + (30 \times 20) \\ &\quad + (7 \times 20) \\ &= 18000 + 600 + 140 \\ &= 18740. \\ 937 \times 5 &= (900 \times 5) + (30 \times 5) + (7 \times 5) \\ &= 4500 + 150 + 35 \\ &= 4685. \end{aligned}$$

3) Therefore, 937 × 25 = 18740 + 4685
= 23425.

From this example it is clear that the "1874" is written one space to the left because it actually represents 18740. The "0" is omitted because it does not—except for indicating placement of "1874"—enter into the final sum (remember the Property of Zero for Addition).

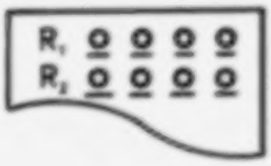
With these problems of recording solved, your flow chart for multiplication should now be in a form usable by anyone who knows some basic multiplication facts. As a test, you might try it out with someone who has never seen or has forgotten the usual algorithm.

0.12 Exercises

1. What would the input card for your multiplication flow chart be if you were asked to compute 758 × 65? The output card?
2. Will your flow chart handle 87 × 9? If so, what input and output cards would be used. If not, why not?
3. If the user of your flow chart knows no multiplication facts at all, how large a table must he be given?
4. Which of the computations below are correct? For those that are incorrect, explain the error made and give the correct answer.

$$\begin{array}{r} 382 \\ \times\ \underline{16} \\ 2292 \\ \underline{382} \\ 2674 \end{array} \qquad \begin{array}{r} 521 \\ \times\ \underline{659} \\ 4589 \\ 2505 \\ \underline{3026} \\ 321139 \end{array} \qquad \begin{array}{r} 85 \\ \times\ \underline{43} \\ \underline{255} \\ \underline{340} \\ 3655 \end{array}$$

5. Above, you might have devised a recorder box like the one below. If so, what would the entries be for 491 × 22?

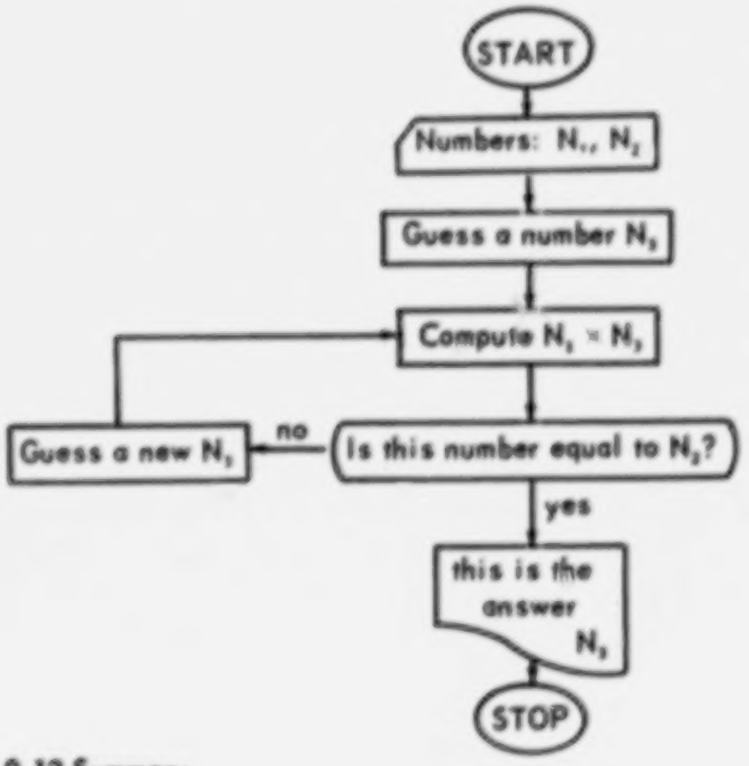


6. One systematic way to divide 750 by 205 is to use subtraction:

$$\begin{array}{rr} & 750 \\ - & \underline{205} \\ & 545 \\ - & \underline{205} \\ & 340 \\ - & \underline{205} \\ & 135 \end{array} \qquad \begin{array}{c} \\ \underline{1} \\ \\ \underline{1} \\ \\ \underline{1} \\ 3 \end{array} \qquad \begin{array}{c}\text{Therefore, } 750 \div 205 = 3 \\ \text{with remainder } 135.\end{array}$$

Try writing a looped flow chart for division using this idea. (Assume the user of your chart knows how to add and subtract.)

7. The process outlined in (6), while suitable for a computer which can perform 1,000,000 subtractions per second, is not too useful for people faced with computations like 57567 ÷ 32 when no machine is available. Will the following flow chart provide _a_ method for quicker dividing? If not, what refinements or corrections can you suggest?



0.13 Summary

At the beginning of this chapter we faced the problem of writing detailed instructions for tasks involving many small steps. To deal with these situations we developed some ideas of flow charting: input and output, process, decision, branching, and looping.

Although the aim of flow charting was to make difficult tasks into ordered sequences of simpler ones, we often found that writing a successful flow chart required thorough understanding of the processes involved. This proved particularly true in our study of addition, subtraction, multiplication, and division of whole numbers. It suggests a good test of comprehension for processes we might encounter in future developments: "Can I write a workable flow chart describing the process?"

0.14 Review Exercises

1. Write out a flow chart showing the steps followed in computing 89 + 63. Be sure that the directions given in your chart state clearly and completely the steps in the process.
2. What type of box might each of the following properly appear in?

(a) red paint.	(e) mud.
(b) krazy kids.	(f) find the product.
(c) the sum is 14.	(g) is the number even?
(d) find the average.	(h) a fast car.

3. What type of box has the form

(a)

(b)

(c)

(d)

4. Describe a typical activity which, when flow charted, would involve looping. Where would the loop or loops enter in?
5. Check the following computations. If they are not correct, explain the source of the error.

$$\begin{array}{r} 7895 \\ +\,\underline{5376} \\ 12161 \end{array} \qquad \begin{array}{r} 8954 \\ -\,\underline{7352} \\ 1602 \end{array} \qquad \begin{array}{r} 456 \\ \times\,\underline{28} \\ \underline{3648} \\ \underline{912} \\ 12768 \end{array} \qquad \begin{array}{r} 873 \\ -\,\underline{942} \\ 69 \end{array}$$

CHAPTER 1: Finite Number Systems

1.1 Jane Anderson's Arithmetic

Mr. Anderson was helping his daughter, who was in first grade, with her arithmetic homework. He asked, "Jane, what is seven plus three?"

Jane looked over her father's shoulder and soon answered, "Seven plus three is ten."

"That is correct," said her father. "Now, what is eleven plus two?"

Jane again glanced over her father's shoulder and said, "Eleven plus two is one."

"My hearing must be bad," said her father. "I thought you said 'Eleven plus two is one.'"

"I did," said Jane.

Her father, of course, wanted to know why she made such a statement. Jane walked over to the clock on the shelf behind her father's shoulder. She explained how she found the sum of 7 and 3. She first pointed to the numeral 7 on the face of the clock and then moved her finger clockwise over 3 numerals. Since she was then pointing at 10 she said, "Seven plus three is ten." Jane proceeded in the same way to find the sum of 11 and 2. She first pointed to the numeral 11 on the clock and then moved her finger clockwise over 2 numerals. Since she was then pointing at numeral 1, she said, "Eleven plus two is one."

1.2 Clock Numbers and Whole Numbers

In answering questions relating to time we probably all have performed an operation quite similar to Jane's procedure. The numbers represented on the face of a clock are the elements of the familiar set of clock numbers:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

If we compare this set with the set of whole numbers:

$$W = \{0, 1, 2, 3, 4, \ldots\}$$

we note one immediate difference. The set of whole numbers is endless or infinite. (That is what the "..." indicates.) If a set is not infinite, we say that it is finite.

The set of numbers that Jane Anderson was using when she said that $11 + 2 = 1$, is a finite number system. Such finite systems have many interesting properties and applications. As we study the clock and other finite number systems, be on the lookout for similarities and contrasts between these systems and the familiar $(W, +)$. Feel free to make guesses or conjectures about properties that appear familiar or unusual. You will find both if you are alert.

1.3 Clock Arithmetic

If you are asked what time it is three hours after seven o'clock, you would naturally answer ten o'clock. We could express this result using the notation "$7 + 3 = 10$". But what if you were asked what time it is two hours after eleven o'clock? Now the answer is one o'clock, and using the same notation as above we would have

$$11 + 2 = 1.$$

In $(W, +)$ it makes sense to assign 13 as the sum of 11 and 2, but on a clock it makes sense to assign 1 as this sum. Similarly, to express the fact that nine hours after seven o'clock is four o'clock, we shall write

$$9 + 7 = 4.$$

Question: On a clock, what is expressed by "$11 + 6 = 5$"?

What if you were asked what time it is seven hours after eight o'clock? Is this the same as finding the sum of 8 and 7 using the arithmetic on a clock?

Question: What is the sum of 8 and 7 using the numbers on a clock? Explain how you obtained your answer.

One approach to answering the above question would be to first place a pointer on a clock face with the pointer directed at "12". In order to compute $8 + 7$, move the pointer clockwise through 8 intervals and then follow this by moving the pointer clockwise through 7 intervals. The pointer will then be directed at "3". Thus 3 is the number assigned as the sum of 8 and 7. (see figure 1.1)

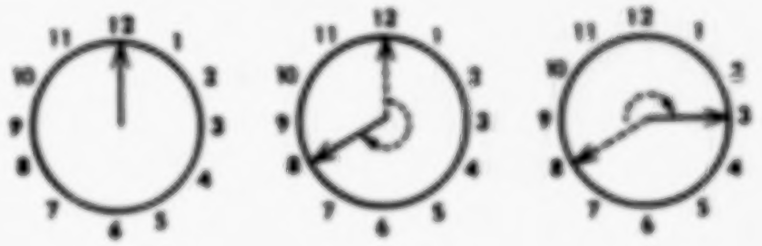


Figure 1.1: Using a dial to determine $8 + 7$.

Let us represent the set of clock numbers, $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ by the symbol Z_{12}. (The Z is suggested by the German word for number, zahlen, and we use the subscript 12 to indicate the the number of elements in this set.)

1.4 Exercises

1. Compute the following sums by the procedure used above.

(a) 9 + 4
(b) 7 + 9
(c) 7 + 8
(d) 5 + 6
(e) 10 + 11
(f) 11 + 10
(g) 1 + 12
(h) 12 + 1
(i) 11 + 11
(k) 12 + 9
(k) 9 + 12
(l) 12 + 12

2. From your computations above, try to make some conjectures concerning properties of addition of clock numbers.
3. Can you find an additive identity element for Z_{12}?
4. What can you conjecture concerning the commutativity of addition in Z_{12}?
5. Are we justified in calling $(Z_{12}, +)$ an operational system? You remember from Chapter 0 that we must decide if for any given ordered pair of elements of Z_{12} we can assign in a natural way an element of Z_{12} to this pair as a sum. One convenient way to settle this question is to construct an addition table for Z_{12}. To indicate that the sum of 11 and 2 is 1, we place "1" in the cell determined by row 11 and column 2. (see Figure 2) Examine the table and note how the sum of 7 and 3 is entered in the table.

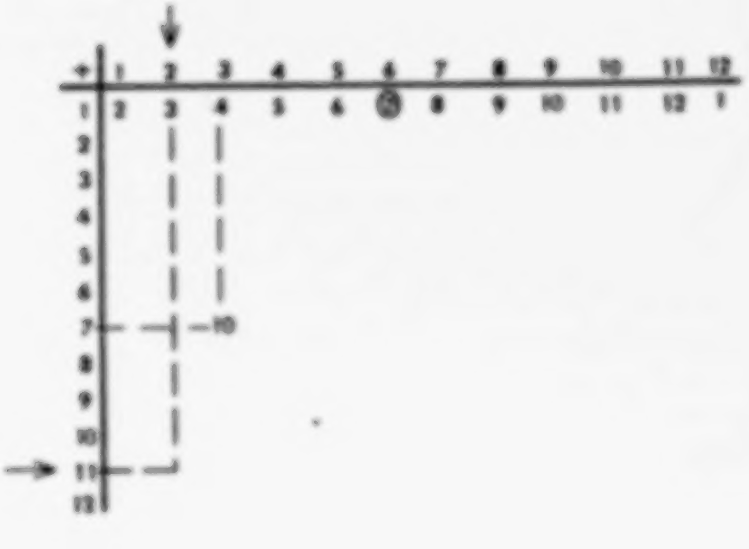


Figure 1.2: Addition Table for Z_{12}

(a) Does the encircled 7 in the body of the table represent 1 + 6 or 6 + 1? Explain your answer.
(b) Discuss why the cells in row 1 were assigned the sums shown in Figure 2.
(c) Copy the table in Figure 2 and compute the entries for the second row, third row, etc. Do you notice any pattern emerging? Can you make any conjectures that can be tested?
(d) What is interesting about the first column? the last column? the last row?
(e) Complete the table. Explain why or why not $(Z_{12}, +)$ is an operational system?
(f) Can you state a difference between the addition table constructed above and an "addition table" for $(W, +)$?
(g) Is addition commutative in Z_{12}?
(h) How can the table provide evidence in support of your answer to (g)?
(i) Is there an additive identity element for $(Z_{12}, +)$?
(j) Can you find another additive identity element for $(Z_{12}, +)$?

6. Make a detailed comparison between $(Z_{12}, +)$ and $(W, +)$ by listing similarities and differences between these two systems.
7. Some common examples of finite sets are:
   (a) the set of vowels in the English alphabet,
   (b) the set of words in a dictionary,
   (c) the set of all sentences which have ever been written,
   (d) the set of clock numbers.

   It is important that you do not confuse a large finite set, such as (c) above, with an infinite set.

   Question 1: Give some examples of large but finite sets.
   Question 2: What, besides W, would be an example of an infinite set?
8. The set W does not contain a greatest whole number.
   (a) How might you argue against someone asserting that he knew the greatest whole number?
   (b) What is the largest finite set you can describe?
9. A googol is a number represented by writing down the numeral 1 followed by a hundred "zeros".
   (a) Is a googol a whole number?
   (b) If we place a pointer on a clock at 12 and move this pointer a googol number of intervals in a clockwise direction, which clock numeral would the pointer then be directed at?

1.5 Calendar Arithmetic

The traffic manager of the nation-wide Bee-Line Trucking Company of New York City was faced with the following situation. Trucks would return to New York City after extended road trips around the country and the manager had to arrange for garage space, the hiring of loaders and extra drivers, service on the truck engines, cargo assignment, etc. The manager found that he needed a fast way of determining the day of the week a truck would return if he knew (1) the day of the week that the truck left New York City and (2) the number of days that the truck would be on the road.

A typical problem was the following: A truck was to leave New York City for Indianapolis, Indiana (2 days); go on to Dallas, Texas (3 days); then to Washington, D.C. (4 days); and finally return to New York City (1 day). If this truck was to leave New York City on Friday, then on what day of the week would it return?

The manager soon struck on the idea of using a dial with days of the week assigned to numbers as in Figure 1.3.

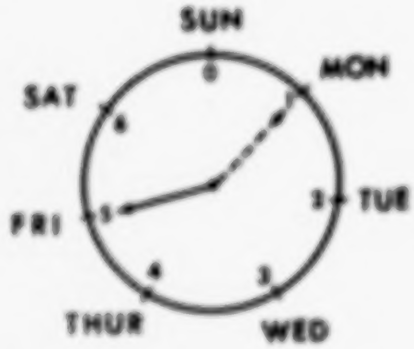


Figure 1.3 Calendar Numbers

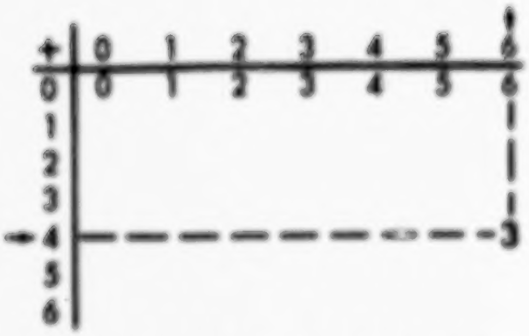
To solve the above problem he proceeded as follows. Since the truck left on Friday, he set the pointer at "5". Since the total trip took 10 days (check this!) he then moved the pointer clockwise through ten intervals. The pointer then was directed at the number 1, so he concluded the truck would return on Monday.

The set of numbers used by the manager, {0, 1, 2, 3, 4, 5, 6} we will call the calendar numbers. We will refer to this set as Z_7.

Consider the following easy problem. If a truck left New York City on a Thursday and returned six days later, then on what day of the week would it return? This problem can be interpreted as asking "What number in Z_7 should be assigned as the sum of 4 (the number associated with Thursday) and 6 (the time of the trip)? We see that the sum obtained from use of the dial agrees with the obvious answer to the original problem, namely Wednesday. Thus in Z_7 we have that $4 + 6 = 3$.

1.6 Exercises

1. Compute the following in Z_7
   - (a) $6 + 1$
   - (b) $2 + 6$
   - (c) $3 + 5$
   - (d) $4 + 2$
   - (e) $4 + 5$
   - (f) $5 + 4$
   - (g) $5 + 5$
   - (h) $5 + 6$
   - (i) $6 + 6$
   - (j) $0 + 1$
   - (k) $0 + 6$
   - (l) $1 + 6$
   - (m) $2 + 5$
   - (n) $5 + 2$
   - (o) $0 + 0$
2. Construct an addition table for Z_7. Note how $4 + 6 = 3$ is recorded in the table.



+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1							
2							
3							
4							3
5							
6							

Figure 1.4: Addition Table for Z_7

   - (a) Explain why, after completing the table, we can dispense with the dial.
   - (b) Is (Z_7, +) an operational system? Explain your answer.
   - (c) Is there an identity element for (Z_7, +)? Explain your answer.
   - (d) Is addition commutative in (Z_7, +)? Why or why not?
   - (e) What interesting patterns can you identify in the rows and columns of the table for (Z_7, +)?
   - (f) What is interesting about the diagonal elements of the addition table for (Z_7, +)? What, if any, patterns can you find?
   - (g) Compare row, column, and diagonal properties of (Z_7, +) and (Z_{12}, +). (You might now replace "12" in Z_{12} by "0".)
3. Compare (Z_7, +) and (W, +). How are they alike? How different? Compare (Z_7, +) and (Z_{12}, +). How are they alike? How different?
4. Consider the following data that the manager of the Bee-Line Moving Company has obtained for one of his routes:

Depart	Arrive	Time of Travel (days)
N.Y.C.	Cleveland, Ohio	2
Cleveland	Jacksonville, Fla.	3
Jacksonville	Atlanta, Ga	1
Atlanta	El Paso, Tex	5
El Paso	Des Moines, Iowa	4
Des Moines	Chicago, Ill.	1
Chicago	N.Y.C.	3

Assume that a truck leaves New York City on a Wednesday.

   - (a) On what day of the week will it arrive in Jacksonville? In El Paso?
   - (b) On what day of the week will the truck return to New York City?

(c) If there is a two-day lay over in El Paso, on what day will it return to new York City?

(d) If a truck leaves on a Saturday, makes the complete route, lays over in New York City for two days, and then makes a second complete route, on what day of the week will it return to New York City?

1.7 Open Sentences

How could you contrast the following mathematical sentences?

$$2 + 3 = 5 \qquad (1)$$

$$5 + 6 = 17 \qquad (2)$$

$$11 + 2 = \square \qquad (3)$$

It is obvious that sentence (1) is true in (W, +) and that sentence (2) is false in (W, +). However, we don't know whether (3) is true or false until the " $\square$ " is replaced by a symbol for a number. Both (1) and (2) are called mathematical statements since they are mathematical sentences that are either true or false (but not both).

Sentence (3) above, and others like it which contain a variable, appear frequently in mathematics. When we say that " $\square$ " is a variable in (3), we mean that the " $\square$ " can be replaced by a symbol for a number from a particular set of numbers. This set of numbers we call the domain of the variable.

If the domain of our variable is Z_{12}, then we could replace " $\square$ " in (3) by "1" and obtain a true sentence

$$11 + 2 = 1.$$

However, if the domain of our variable is W, then replacement of " $\square$ " by "1" would yield a false sentence. To obtain a true statement in W we should replace " $\square$ " by "13" since

$$11 + 2 = 13.$$

In dealing with sentences such as (3), always be aware of the domain of the variable(s) which you are considering.

A sentence such as "$11 + 2 = \square$", which is neither true nor false, is called an open sentence. Note that such a sentence will become either true or false after replacement of the box. It is easy to write down open sentences, that is, sentences which contain at least one variable and which are neither true nor false. Examples of open sentences are

$$\square + 2 = 6,$$

$$3 + 4 = \triangle,$$

$$7 + \square = 11,$$

$$\triangle + \square = 4,$$

$$\text{and} \quad \square = 0.$$

An open sentence, like those above, with the equality sign is called an equation. Another kind of sentence used frequently in mathematics deals with the inequality relations "is less than" and "is greater than". For example, in the set of whole numbers we can write such sentences as "3 is less than 4" and "8 is greater than 6". We use the symbols "<" and ">" to denote, respectively, "is less than" and "is greater than". Thus we can rewrite the above sentences as "$3 < 4$" and "$8 > 6$". Examples of open sentences using these relations are

$$5 > \square + 1,$$

$$4 < \triangle + 6,$$

$$\text{and} \quad \square > 0.$$

An open sentence with an inequality symbol is called an inequation or an inequality.

Frequently you will be asked to solve an open sentence. This means that you are to determine those numbers which, when substituted for the variables, yield true statements. The set of numbers which yield true statements is called the solution set of the open sentence.

Question 1: Why is {0} the solution set of the open sentence $\square + 4 = 4$, where the domain of " $\square$ " is W?

Question 2: What is the solution set of the open sentence $\triangle + 3 = 1$, if the domain of "$\triangle$" is Z_{12}?

Question 3: What is the solution set for the open sentence $2 \times \square = 5$, where the domain of " $\square$ " is W?

You have probably already determined that the solution set for the open sentence in Question 3 has no members. It is an example of an empty or null set. Some other empty sets are the set of all men who are thirty feet tall and the set of all whole numbers between 1/2 and 3/4. We usually indicate empty sets by the symbols "∅" or "{ }".

Question 4: Discuss why the solution set for Question 1, $\{0\}$, is not the same as the solution set for Question 3, $\emptyset$.

We have been using "$\square$" and "$\triangle$" to denote variables. It is more usual in mathematics to denote a variable by such a symbol as x, y, z, or n. If we use these symbols to rewrite the examples of open sentences given earlier, they would be

$$x + 2 = 6,$$
$$3 + 4 = y,$$
$$5 > n + 1,$$
and
$$x + y = 4.$$

Let us review some of the above ideas by considering the following.

Example 1. Let the domain of the variable be W. If we are asked to solve $\square + 5 = 12$ and $\square$ is replaced by 7, we obtain $7 + 5 = 12$, which is true. hence 7 is a solution of the open sentence, or $\{7\}$ is the solution set, since no other replacement from W would give a true statement.

Example 2. Let the domain of the variable be Z_7. In the open sentence $\square + 4 = 3$, if $\square$ is replaced by 6, we obtain $6 + 4 = 3$ which is true. Hence 6 is the solution or $\{6\}$ is the solution set, since no other replacement of $\square$ make the sentence true.

Example 3. Let the domain of the variable be W. For the open sentence $\square + 4 = 3$, we find that there is no replacement from W which yields a true sentence. Hence there is no solution in W or the solution set is the empty set.

Example 4. Let the domain of the variable be Z_{12} with "12" replaced by "0". That is $Z_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. In $x + 4 = 3$, if x is replaced by 11, then we obtain $11 + 4 = 3$ which is the true in Z_{12}. Hence 11 is a solution to the open sentence, or $\{11\}$ is the solution set, since no other replacement from Z_{12} would give a true statement.

Example 5. If the domain of the variable is W, then the solution set of the inequation n $n + 3 < 8$ is $\{0, 1, 2, 3, 4\}$.

Example 6. If the domain of the variable is W, then the solution set of $y + 3 > 8$ is the set of whole numbers greater than 5, that is, $\{6, 7, 8, 9, \ldots\}$

1.8 Exercises

1. Which of the following are true senteaces in $(W, +)$

   (a) $2 + 3 = 4 + 1$

   (b) $3 + 4 > 5 + 2$

   (c) $\square + 2 = 5$

   (d) $\square = \square + 1$

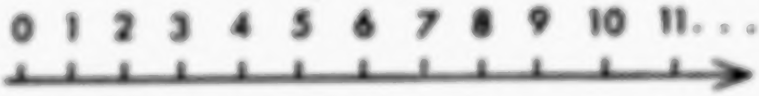
   (e) $\square + 0 = \square$

2. Explain why or why not the following are true sentences.

   (a) $11 + 7 = 5$ in $(Z_{12}, +)$

   (b) $5 + 5 = 3$ in $(Z_7, +)$

3. Below is a number line where the first twelve whole numbers are used as labels for points spaced one-half inch apart.



To help you answer the following questions you should draw a line on a sheet of paper similar to the one above.

(a) Is 5 less than 9?
(b) Is the point labeled "5" to the left of the point labeled "9"?
(c) 11 is __?__ than 3; point 11 is to the __?__ of point 3.
(d) 61 is __?__ than 63; point 61 is to the __?__ of point 63.
(e) If x and y are whole numbers, then $x < y$ means the point for x is to the ________ of the point for y.
(f) If x and y are whole numbers, then $x > y$ means the point for x is to the ________ of the point for y.
(g) Using "less than" and "greater than" state a general test for deciding when a whole number is between two given whole numbers.
(h) If x is any whole number, what can one say about $x + 1$?
(i) If x, y, and z are whole numbers and $x < y$ and $y < z$, then what can you conclude?

(j) If x, y, and z are whole numbers and $x < y$ and $x < z$, then what can you conclude?

(k) If $y > x$ and $z < n$ and $z > y$, then what can you conclude?

(l) If x and y are any two whole numbers and $x \neq y$, then what can you conclude concerning their respective positions on the line?

5. Solve the following open sentences–given that the domain of the variable is W. (If the same symbol for a variable appears more than once in a sentence, then it represents the same number each time that it occurs.)

(a) $\square + 7 = 15$

(b) $\square + \square = 6$

(c) $13 = \square + 1$

(d) $7 + x = 15$

(e) $35 + y = 25$

(f) $z + 1 > 3$

(g) $\square + 1 > 1$

(h) $n < 4$

(i) $x + 1 < 11$

(j) $x + x + 1 = 11$

6. Solve the following open sentences–given that the domain of the variable is Z_7.

(a) $2 + 5 = \square$

(b) $\square + 3 = 3$

(c) $\square + 4 = 1$

(d) $6 + x = 1$

(e) $6 + 6 = y$

(f) $\square + \square = 5$

(g) $\square + 1 > 1$

(h) $Z + 1 > 3$

7. Solve the following open sentences–given that the domain of the variable is Z_{12}.

(a) $5 + 8 = \square$

(b) $4 + 3 = x$

(c) $y + 3 = 9$

(d) $9 + z = 2$

(e) $7 + x = 7$

(f) $x = x + 1$

8. Using the symbol "x", write an open sentence whose solution set is $\{6\}$ where the domain of the variable is

(a) W (b) Z_7 (c) Z_{12}.

9. Using the symbol "x", write an open sentence whose solution set is

(a) W (b) the empty set

10. Explain why or why not " $\square + 3 = 4$" is

(a) a sentence
(b) an open sentence
(c) a statement
(d) an equation whose solution set is $\{1\}$ if the domain of the variable is Z_7.

11. If two open sentences have the same solution set then we say they are equivalent.

Find the solution set of each pair of sentences below. Then determine if the sentences are equivalent. Let the domain be W.

(a) $12 + \square = 31$: $\square + 7 = 36$

(b) $x + 3 = 3$: $4 + x = 2$

(c) $z + 2 < 5$: $8 > z + 5$

(d) $x = 3 + 0$: $0 + 3 = x$

(e) $y = 2 + 0$: $y + 2 = 0$

1.9 New Clocks

In 1.3 we constructed a "clock arithmetic" for the familiar clock which uses twelve numerals on its face. If we wanted a clock with exactly seven numerals on its face, one possibility would be to use the system constructed in 1.5. There, for the calendar numbers, we made use of a pointer and the set $\{0, 1, 2, 3, 4, 5, 6\}$. We have already explored how the arithmetic for such a 7-clock would proceed. (See 1.6 Exercises)

Our previous work with a 12-clock and a 7-clock indicates that we could easily construct other clocks. For example, a 4-clock is suggested by a lamp switch which has four positions: "Off", "Low", "Medium", and "Bright". If we assign, in order, the numerals "0", "1", "2", and "3" to these four positions and again introduce a pointer, we can draw a picture of such a 4-clock.

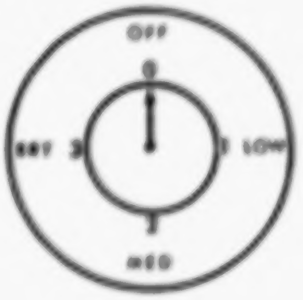


Figure 1.5

1.10 Exercises

1. (a) Construct an addition table for the 4-clock.

(b) Use your addition table to do these computations.

(1) $1 + 2$ (3) $2 + 1$ (5) $2 + 2$ (7) $1 + 3$
(2) $3 + 1$ (4) $3 + 3$ (6) $0 + 3$ (8) $3 + 2$

(c) If we denote the set $\{0, 1, 2, 3\}$ by Z_4, is $(Z_4, +)$ an operational system?

(d) Use your table to determine the position a lamp switch would be in if, starting from the "Off" position we turned it clockwise through 3 intervals and followed this with another clockwise turn through 2 intervals.

   (e) How can an examination of the table help you decide whether or not addition in Z_4 is commutative?

2. (a) Compare $(Z_4, +)$ with $(Z_7, +)$.
   (b) Compare $(Z_4, +)$ with $(W, +)$.
3. (a) Make up an addition table for a 5-clock using the set $Z_5 = \{0, 1, 2, 3, 4\}$
   (b) Is $(Z_5, +)$ an operational system?
   (c) Using your addition table for Z_5 compute the following:

(1) 2 + 4	(3) 3 + 2	(5) 3 + 3
(2) 1 + 4	(4) 4 + 4	(6) 4 + 3

   (d) Compare $(Z_5, +)$ with $(Z_4, +)$.
   (e) A burner on an electric stove is controlled by a circular switch. The five possible positions are arranged and labeled in the following order: "Off", "Simmer", "Low", "Medium", and "High". Make up three problems which the table constructed in 3 (a) can help you solve (see Exercise (c) above).
4. (a) What kind of a clock is suggested by the channel selector knob on a T.V. set?
   (b) Can you find any everyday applications of any clocks that have been, or could be, constructed?
5. Examine the tables for addition of clock numbers that you have constructed.
   (a) What properties of addition tables can you find which make them easy to construct?
   (b) Make up an addition table for $Z_6 = \{0, 1, 2, 3, 4, 5\}$. Why is $(Z_6, +)$ an operational system?

   NOTE: Keep this table for $(Z_6, +)$, and all other tables that you construct, for future use.

1.11 Rotations

In Figure 1.6 a six sides geometric figure called a regular hexagon is drawn in a circle with center point labeled C. We say that C is also the center of the regular hexagon. The points of the hexagon which are on the circle are called the vertices of the regular hexagon and are labeled 0, 1, 2, 3, 4, and 5.

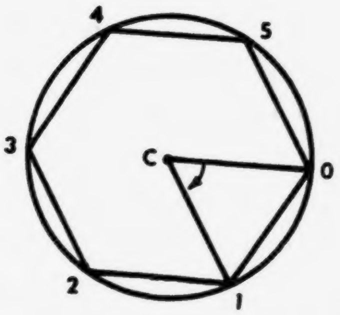


Figure 1.6: A regular hexagon inscribed in a circle

If we keep the center C of the hexagon fixed and revolve the hexagon in a clockwise direction until the vertex at "0" is moved to "1", and the vertex at "1" is moved to "2", etc. then we say that we have a rotation of the regular hexagon through 60° about C. Let us denote this rotation by r_1 ("r" to suggest rotation and "1" to suggest 1 interval of 60°). The point C is called the center of rotation. If our rotation about C passes through 120° (or 2 intervals) we shall denote this by r_2. Another way to view r_2 is as the result of performing an r_1 rotation and then following it with another r_1 rotation. What would r_3 denote? What would r_4 denote? etc. r_6 is of particular interest since then we would be in what position?

Let us examine what r_5 denotes. We could say that in r_5 we have an instruction to rotate the regular hexagon about C through five intervals or 300°. If we followed the r_5 instruction by the r_1 instruction, then we would have completed a rotation of 360° which is, of course, one complete rotation. If we examine the subscripts of r_5 and r_1 we find that r_6 is suggested as the result of following r_5 by r_1. But r_6, considered alone, means that we are once again at our original position. In other words, the instructions r_6 and r_0 have the same effect. We choose to call such an instruction "r_0". We say that r_0 is the result of following r_5 by r_1 or that r_0 is assigned to r_5 and r_1.

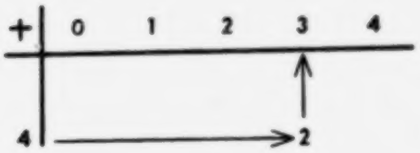
1.12 Exercises

1. What result would you assign to the following!
   (a) r_1 followed by r_2
   (b) r_1 followed by r_3
   (c) r_2 followed by r_5
   (d) r_4 followed by r_2
   (e) r_3 followed by r_0
2. (a) Why was r_0 said to be the same instruction as r_6?
   (b) What is the result if any instruction is followed by r_0?
   (c) What special name might be given to r_0? Justify your answer.
3. Examine the partially completed table below.

	r_0	r_1	r_2	r_3	r_4	r_5
r_0						r_5
r_1		r_2				
r_2					r_0	
r_3						
r_4						
r_5			r_1			

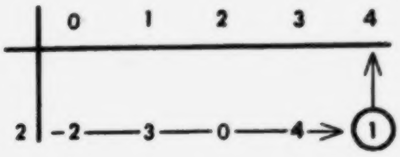
Is subtraction an always possible process in the set of clock numbers?

In particular, let us determine if "2 – 4" names a number in the set Z_5. Our experience with subtraction in the whole numbers suggests that we agree to the following: If there is one and only one number in Z_5 which when added to 4 yields 2 we shall say that "2 – 4" names this number. In order to see that such a number does exist, we can make use of the addition table for (Z_5, +). First locate "4" at the left in the table for (Z_5, +). (See the partially completed table below.) Move across this row until you find the entry "2". What is the number heading the column in which we find "2"? We find a "3".



We conclude that $2 - 4 = 3$ in Z_5 because 3 is the number in Z_5 which when added to 4 yields 2.

Example 1. In order to find the number named "1 – 2" in Z_5 we seek the number in Z_5 which when added to 2 yields 1. From the table for (Z_5, +) we see that $2 + 4 = 1$



Thus we conclude that $1 - 2 = 4$ in Z_5. Note that 4 is the only number in (Z_5, +) which when added to 2 yields 1. Thus the difference 4 is unique. By this we mean there is one and only one number, 4, which when added to 2 yields 1.

Example 2. In order to find the number named by "4 – 1" in Z_5 we seek the one and only one number which when added to 1 yields 4.

Since $1 + 3 = 4$ in (Z_5, +) we conclude that $4 - 1 = 3$ in Z_5. Do you see that the difference 3 is unique?

Example 3. In order to compute $0 - 3$ in Z_5 we seek the one and only one number which when added to 3 yields 0 in (Z_5, +).

Since $3 + 2 = 0$ in (Z_5, +) we conclude that $0 - 3 = 2$ in Z_5.

Note: In many exercises in this chapter we shall use the word "compute" to mean that we are to express the name of a number using one of the clock numerals. In Example 3 above you see that we computed in Z_5. Thus our result would use only the numerals "0", "1", "2", "3", and "4". Such numerals will be referred to as the simplest names for clock numbers. Thus in Example 3 we expressed the result of computing $0 - 3$ in Z_5 by using the simplest name "2". That is $0 - 3 = 2$ in Z_5.

1.14 Exercises

1. Using your addition table for (Z_5, +) find the simplest name for

(a) $1 - 4$
(b) $4 - 3$
(c) $3 - 4$
(d) $4 - 2$
(e) $4 - 4$
(f) $3 - 0$
(g) $0 - 4$
(h) $2 - 3$
(i) $1 - 3$
(j) $0 - 2$

2. Below is a partially completed subtraction table where the results obtained in Examples 1, 2, and 3 have been recorded.

–	0	1	2	3	4
0				2	
1			④		
2					
3					
4		3			

Note that the encircled 4 in the table indicates that $1 - 2 = 4$ in Z_5. (Check Example 1 above).

a) Copy the above table and compute the remaining entries in this subtraction table.
b) Is subtraction an "always possible" process in the set Z_5? Explain your answer.
c) How does the process of subtraction in Z_5 compare with the process of subtraction in W?
d) Explain why, or why not, (Z_5, –) is an operational system.

3. You recall that we introduced addition for our finite sets by making use of a clock. Then we constructed addition tables. However, subtraction in Z_5 was first introduced by using the idea of a table. The following exercises relate this subtraction to a 5 – clock.

a) If the pointer of a 5 – clock is placed on the numeral 2 and then moved counter-clockwise through 3 intervals, then at what numeral is the pointer directed? What subtraction problem in Z_5 does this solve?
b) Can you state how we could find the simplest name for $1 - 2$ on a 5 – clock?
c) Find the simplest name for each of the following and then describe how to check your results using a 5 – clock.

1) $1 - 4$
2) $0 - 3$
3) $4 - 1$
4) $2 - 4$

4. Use your table for $(Z_6, +)$ to find the simplest name for
   a) $5 - 2$
   b) $2 - 5$
   c) $4 - 1$
   d) $1 - 4$
   e) $3 - 4$
   f) $0 - 4$
   g) $2 - 3$
   h) $1 - 5$
5. Use your table for $(Z_7, +)$ to find the simplest name for
   a) $1 - 6$
   b) $5 - 6$
   c) $4 - 6$
   d) $2 - 6$
   e) $6 - 2$
   f) $6 - 0$
   g) $0 - 6$
   h) $6 - 6$
   i) $3 - 5$
   j) $3 - 6$
   k) $0 - 0$
   l) $1 - 4$
6. Find an identity element, if one exists, in
   a) $(Z_5, -)$
   b) $(Z_6, -)$
   c) $(Z_2, -)$
7. What generalization is suggested by Exercise 6 above?
8. Determine if subtraction is commutative in the following:
   a) $(Z_5, -)$
   b) $(Z_6, -)$
   c) $(Z_2, -)$
9. What generalization is suggested by Exercise 8 above?
10. Solve the following open sentences given that the domain of the variable is Z_5.
    a) $2 - 4 = x$
    b) $y - 4 = 1$
    c) $3 - z = 1$
    d) $3 - x = 2$
    e) $1 - 4 = y$
    f) $2 - 3 = z$
    g) $3 - x = 3$
    h) $3 - y = 4$
    i) $3 - z = 0$
    j) $0 - x = 0$
    k) $1 - 3 = y$
    l) $z - 4 = 4$
11. Solve the following open sentences where the domain of the variable is Z_6.
    a) $3 - 5 = x$
    b) $2 - 5 = y$
    c) $1 - 2 = z$
    d) $0 - x = 2$
    e) $5 - 2 = y$
    f) $y - 4 = 4$
    g) $z - 4 = 5$
    h) $1 - x = 3$
    i) $0 - y = 2$
    j) $0 - z = 0$

1.15 Multiplication in Clock Arithmetic

We shall now consider how multiplication will be defined in clock arithmetic. From your previous study of the whole numbers, you know that given the pair of whole numbers 3 and 4 you would assign the whole number 12 to this pair as their product. In short, $3 \cdot 4 = 12$ in $(W, \cdot)$.

But how should we define the product $3 \cdot 4$ in, for example, Z_5? Even though we could assign any number in Z_5 as this product, let us agree that the product $3 \cdot 4 = 2$ in Z_5. Why we select 2 as the product can be seen if we note the following relationship between $(W, \cdot)$ and the 5-clock. In $(W, \cdot)$ we have that $3 \cdot 4 = 12$. If we place the pointer of a 5-clock on "0" and then move it clockwise through 12 intervals the pointer will be directed at "2". Using this result we define $3 \cdot 4 = 2$ in Z_5.

We shall use the above relationship between $(W, \cdot)$ and the 5-clock to define $2 \cdot 4$ in Z_5. Since $2 \cdot 4 = 8$ in $(W, \cdot)$ we move the pointer of a 5-clock clockwise through 8 intervals from "0". The pointer is then directed at "3". Using this result we define $2 \cdot 4 = 3$ in Z_5.

How should we define $4 \cdot 4$ in Z_5? Since $4 \cdot 4 = 16$ in $(W, \cdot)$ we move the pointer of a 5-clock clockwise through 16 intervals from "0". The pointer is then directed at "1". Thus we define $4 \cdot 4 = 1$ in Z_5.

There is another approach to multiplication in Z_5 which does not use the idea of a clock. The key idea in this second approach is that of remainder. Do you remember how this term was used in your earlier study of mathematics? For example, if the whole number 8 is divided by the whole number 5 we obtain a quotient of 1 and a remainder of 3. Recall that in defining $2 \cdot 4$ in Z_5 we moved a pointer on a 5-clock through 8 intervals and the pointer was directed at "3". We see that in this example the resulting product given by use of the clock is precisely the remainder obtained when 8 is divided by 5. Will the remainder approach continue to give results equivalent to the clock approach?

We can test to see if the products defined earlier on pairs of numbers in Z_5 are related to "remainders". For example, earlier we defined $4 \cdot 4$ to be 1 in Z_5. By the remainder approach we first note that $4 \cdot 4 = 16$ in $(W, \cdot)$. Then we divide 16 by 5 and obtain a quotient of 3 and a remainder of 1. If we disregard the quotient and examine the remainder we see that this remainder, 1, is the same number which we defined earlier as the product of 4 and 4 in Z_5.

If we apply this remainder approach in order to define $3 \cdot 4$ in Z_5, we proceed as follows: Compute $3 \cdot 4$ in $(W, \cdot)$ and obtain 12; Divide 12 by 5 obtaining a quotient of 2 and a remainder of 2; we record the remainder 2 as the product; Thus $3 \cdot 4 = 2$ in Z_5. Again this agrees with an earlier result.

Let us now indicate a scheme whereby we can assign a "product" to any pair of numbers in Z_5. If a and b are any two numbers in $Z_5 = \{0, 1, 2, 3, 4\}$, we first form their product in $(W, \cdot)$. This product is then divided by 5, and we note the remainder. From above, we know that this remainder is also a number in Z_5. We record this remainder and call it the "product of a and b", which we write as "$a \cdot b$".

Examples: (1) If we wish to compute the product of 3 and 3 in Z_5 we note first that $3 \cdot 3 = 9$ in $(W, \cdot)$. When 9 is divided by 5 we obtain a quotient of 1 and a remainder of 4. We disregard the quotient and record the remainder 4 as the product we are seeking. Thus,

$$3 \cdot 3 = 4 \text{ in } (Z_5, \cdot)$$

(2) The product of 2 and 2 in $(Z_5, \cdot)$ is found by noting that $2 \cdot 2 = 4$ in $(W, \cdot)$ and 4 divided by 5 yields a quotient of 0 and a remainder of 4. We disregard the quotient and record the remainder 4 as the product. Thus

$$2 \cdot 2 = 4 \text{ in } (Z_5, \cdot)$$

(3) The product of 3 and 0 is 0 in Z_5 because $3 \cdot 0 = 0$ in $(W, \cdot)$ and 0 divided by 5 yields a remainder of 0.

1.16 Exercises

1. Below is a partially completed multiplication table for pairs of numbers in Z_5. Some of the products obtained above have been recorded.

•	0	1	2	3	4
0					
1					
2			4		3
3	0			4	2
4					1

a) Copy the above table and compute the remaining entries.

b) Explain why $(Z_5, \cdot)$ is an operational system.

c) Examine the second row and second column of the table above. What is true when a number in Z_5 is multiplied by 1? What is true when 1 is multiplied by any number in Z_5? What property of $(W, \cdot)$ does this suggest? Can you state a corresponding property for $(Z_5, \cdot)$?

d) How does an examination of the above table provide evidence of a commutative property for $(Z_5, \cdot)$. Explain why, or why not, multiplication is commutative in $(Z_5, \cdot)$.

e) Examine the entries in the first row. Do they suggest a property of $(W, \cdot)$ which might also hold in $(Z_5, \cdot)$? Explain.

f) Examine the entries in the first column. What property do these entries suggest for $(Z_5, \cdot)$?

g) Note that the entries in the multiplication table for $(Z_5, \cdot)$ are all numbers from the set $Z_5 = \{0, 1, 2, 3, 4\}$. Why is this true? Consider what possible remainders you can obtain when a whole number is divided by 5?

2. Solve the following open sentences in $(Z_5, \cdot)$

(a) $3 \cdot x = 1$ (e) $3 \cdot x = 2$ (i) $3 \cdot x = 3$

(b) $4 \cdot x = 4$ (f) $3 \cdot x = 0$ (j) $4 \cdot z = 1$

(c) $y \cdot 2 = 0$ (g) $4 \cdot x = 2$ (k) $3 \cdot y = 4$

(d) $0 \cdot x = 0$ (h) $1 \cdot y = 3$ (l) $0 \cdot x = 2$

3. If we wish to construct a multiplication table for $(Z_4, \cdot)$ we record remainders resulting from division by 4. Thus, to determine the product of 2 and 3 in $(Z_4, \cdot)$ we note that 6, the product of 2 and 3 in $(W, \cdot)$, when divided by 4, gives quotient 1 and remainder 2. Thus,

$$1 \cdot 3 = 2 \text{ in } (Z_4, \cdot)$$

(a) Copy and complete the following (multiplication) table for $(Z_4, \cdot)$

•	0	1	2	↓ 3
0				
1				
→ 2				2
3				
4				

(b) What are the entries in the first row? What property do these entries suggest for $(Z_4, \cdot)$?

(c) What are the entries in the second row? What property do these entries suggest for $(Z_4, \cdot)$?

(d) Examine carefully the entries in the third row. Can you discover a property which holds in $(W, \cdot)$ and which fails to hold for $(Z_4, \cdot)$? Does this property hold for $(Z_5, \cdot)$?

4. Construct multiplication tables for $(Z_6, \cdot)$ and for $(Z_7, \cdot)$.

(a) Examine the tables for $(Z_4, \cdot)$ and $(Z_6, \cdot)$. In what ways are these tables similar? In what ways are they different?

(b) Examine the tables for $(Z_6, \cdot)$ and $(Z_7, \cdot)$. What properties do they have in common? Can you find some essential difference between the tables?

5. The following exercise provides an opportunity to do some research. The tables for $(Z_4, \cdot)$ and $(Z_6, \cdot)$ differ in at least one essential way from

the tables for $(Z_5, \cdot)$ and $(Z_7, \cdot)$. If we disregard the row and column headed by "0" we see that there is no repetition of the entries in the remaining rows and columns. But there is a repetition of some entries in the rows and columns of the tables for Z_4 and Z_6. An interesting problem is to predict what other clock multiplication systems will be of the "$(Z_6, \cdot)$ type" and which of the "$(Z_7, \cdot)$ type". What other tables besides those for $(Z_4, \cdot)$ and $(Z_6, \cdot)$ have the "repetition of entries" property? Experiment by examining the tables for $(Z_3, \cdot)$ and $(Z_8, \cdot)$.

(a) Does the $(Z_3, \cdot)$ table behave as the "$(Z_6, \cdot)$ type" table or as the "$(Z_7, \cdot)$ type" table with regard to repetition of entries?

(b) Does the $(Z_8, \cdot)$ table behave as the "$(Z_6, \cdot)$ type" table or as the "$(Z_7, \cdot)$ type" table?

(c) Can you detect any pattern developing? Can you make a conjecture concerning which tables behave as did the $(Z_6, \cdot)$ table and which behave as did the $(Z_7, \cdot)$ table?

(d) After you have a conjecture, test it out by considering the multiplication table for $(Z_9, \cdot)$. Does your conjecture still hold true?

(e) You might want to experiment further in order to find a pattern that predicts how clock multiplication tables will behave as regards "repetition of the entries." Can you predict which elements in a given clock number system will repeat as entries in the multiplication table?

6. Write a flow chart which describes the algorithm for finding products in $(Z_7, \cdot)$.

Note: We need a notation in order to express a general set of clock numbers. You have seen that set $Z_4 = \{0, 1, 2, 3,\}$ has 3 as its "last" element, and that $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$ has 6 as its "last" element. What if a set of clock numbers had m elements? It is clear that the "last" element would be m – 1. Thus, we denote a set of clock numbers containing m elements by

$$Z_m = \{0, 1, 2, \ldots, m-1\}$$

7. Write a flow chart which describes an algorithm for finding products in $(Z_m, \cdot)$.

8. Solve the following open sentences

(a) In $(Z_4, \cdot)$:

(1) $3 \cdot x = 2$ (2) $2 \cdot y = 2$
(3) $2 \cdot z = 0$ (4) $3 \cdot x = 1$
(5) $0 \cdot x = 0$ (6) $0 \cdot z = 3$
(7) $1 \cdot y = 3$ (8) $2 \cdot x = 3$

(b) In $(Z_7, \cdot)$:

(1) $6 \cdot x = 3$ (2) $3 \cdot y = 5$
(3) $5 \cdot y = 1$ (4) $y \cdot 2 = 0$
(5) $4 \cdot y = 2$ (6) $x \cdot 0 = 5$
(7) $6 \cdot z = 5$ (8) $6 \cdot x = 0$

(c) In $(Z_6, \cdot)$:

(1) $5 \cdot x = 3$ (2) $4 \cdot y = 5$
(3) $4 \cdot z = 3$ (4) $4 \cdot x = 0$
(5) $4 \cdot h = 2$ (6) $4 \cdot z = 1$
(7) $3 \cdot x = 3$ (8) $3 \cdot y = 0$
(9) $3 \cdot y = 5$ (10) $2 \cdot z = 2$
(11) $2 \cdot x = 0$ (12) $2 \cdot y = 4$

9. Suppose you wanted to make up a code in order to send a secret message to a friend. One type of code is called a substitution code. In such a code one letter of the alphabet is substituted for another letter by means of a key or by writing some formula which indicates how the substitutions are made. For example, if each letter is replaced by the one that follows it in the alphabet, then we can describe this substitution by the formula

$$x' = x + 1 \text{ in } (Z_{26}, +)$$

This means that any letter x is replaced by the letter x' (read "x prime") which follows it. Thus "b", "k", and "q" would be replaced, respectively by "c", "1", and "r".

(a) How would you encode, that is, put into code, the word "DANGER"?

(b) How would you decode the word "IFMQ"?

(c) Why was Z_{26} used in the above formula?

(d) What would the formula be if "a" is replaced by "d", etc.? If we wished to make our messages more readable, we could include number assignments for such words as "space", "comma", and "period" and then use Z_{29} where originally we assign 0 to a, 1 to b, c, . . . , 23 to x, 24 to y, 25 to z, 26 to "space", 27 to "comma", and 28 to "period". Using the formula $x' = x + 1$ in $(Z_{29}, +)$ then we would encode "JAMES BOND" as

10 1 13 5 19 27 2 15 14 4

(Note that 27 represents the word "space".)

(e) Examine the following coded message. The system (Z_{29}, ·) is used. Can you find the formula that tells how substitutions are made and then decode the message?

(1) 6 14 14 6 14 11 26 8 18 26
19 7 4 26 18 15 24 28

(2) 11 16 0 7 16 9 14 11 21 10 0
22 7 26 22 1 0 22 10 7 0
14 7 22 22 7 20 0 7 0
17 5 5 23 20 21 0
15 17 21 22 0
17 8 22 7 16 2

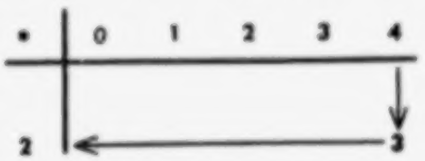
(f) Explain why using a formula such as

$$X' = 2 \cdot x \text{ in } (Z_{26}, \cdot)$$

will not work. What goes wrong? Would the same problem arise if we had used the same formula $x' = 2 \cdot x$ but instead worked in (Z_{29}, ·)?

1.17 Division in Clock Arithmetic

Suppose we wish to divide 3 by 4 in Z_5. Let us recall our experince with division in the whole numbers and use the symbols "3 ÷ 4" or "3/4" to denote such a quotient. We read these symbols as "3 divided by 4" or "3 over 4" and assume they mean the same thing. What follows is suggested by how the division process was carried out in W. Let us agree that to evaluate 3/4 in Z_5 we seek one and only one number in Z_5 which when multiplied by 4 yields 3. The partially completed multiplication table given below indicates how we can search for such a number. First locate "4" at the top of the table. Move down the column headed by this "4" until you find the entry "3". What is the number heading the row in which we find "3"? We find a "2". Thus we conclude that "3/4" and "2" are two names for the same number in (Z_5, ·).



We can express this with the equation 3/4 = 2.
Note that there is one and only one number in Z_5, namely 2, which is equal to 3/4.

Example 1. How do we evaluate 4/3 in Z_5? In order to evaluate 4/3 in Z_5 we proceed as follows. We ask, "Does there exist one and only number in Z_5 which when multiplied by 3 in (Z_5, ·) yields 4?" From our table for (Z_5, ·) We see that 3 · 3 = 4 in (Z_5, ·). Thus we conclude that 4/3 = 3 in Z_5.

Example 2. How do we evaluate 0/3 in Z_5? We seek one and only one number in Z_5 which when multiplied by 3 in (Z_5, ·) yields 0. Since 0 · 3 = 0 in (Z_5, ·) we conclude that 0/3 = 0 in Z_5.

1.18 Exercise

1. Find the simplest names in Z_5 for
   (a) 3/2
   (b) 4 ÷ 2
   (c) 4/1
   (d) 1 ÷ 3
   (e) 4/4
   (f) 2/3
   (g) 2 ÷ 4
   (h) 1/4
   (i) 0/1
   (j) 0/4
   (k) 1/2
   (l) 2 ÷ 1
   (m) 1/3
   (n) 3/1
   (o) 0/0
2. Try to find the simplest names in Z_5 for
   (a) 1/0
   (b) 2/0
   (c) 3/0
   (d) 4/0
3. Explain why or why not (Z_5, ÷) is an operational system.
4. What do you notice when you try to construct a division table for pairs of numbers in Z_5? How does this table compare with the multiplication table for (Z_5, ·)?
5. In Z_6 we have that 4/2 = 5 because 5 · 2 = 4 in (Z_6, ·). Make an investigation of the process of division in other pairs of numbers in Z_6. Write a brief report of your findings.

1.19 Properties of Clock Arithmetic

If we examine the tables given below for (Z_4, +) and (Z_4, ·) certain properties of these operational systems are easily found.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

•	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

For example, there exists an additive identity element, namely 0, in (Z_4, +). There also exists a multiplicative identity element, namely 1, in (Z_4, ·). We find that addition is commutative in (Z_4, +) and that multiplication is commutative in (Z_4, ·). Your earlier work in this chapter should suggest that corresponding properties will hold for any set of clock numbers that we might choose.

Let us now examine a property which may be new to you. First, check that the following equations are true statements in $(Z_4, +)$.

$$0 + 0 = 0 \qquad (1)$$
$$2 + 2 = 0 \qquad (2)$$
$$1 + 3 = 0 \qquad (3)$$
$$3 + 1 = 0 \qquad (4)$$

In each of the above equations we have a pair of numbers whose sum is 0 in $(Z_4, +)$. Or, we could say that we have found pairs of numbers in Z_4 whose sum is the the identity element 0 in $(Z_4, +)$. The numbers in such a pair, whose sum in $(Z_4, +)$ is the identity element, are called inverses of each other under addition in Z_4. The numbers 1 and 3 are inverses of each other since $1 + 3 = 0$ and $3 + 1 = 0$ in $(Z_4, +)$. We also say that 1 is the inverse of 3 and 3 is the inverse of 1. Note that (1) shows that 0 is its own inverse and (2) shows that 2 is its own inverse. (3) and (4) taken together show that 1 and 3 are inverses of each other.

We can search for inverses in any operational system which has an identity element. Thus in $(Z_4, \cdot)$ we will say that a pair of numbers are inverses if their product is the identity element 1. The inverse of 3 is easily obtained by examining the table for $(Z_4, \cdot)$. We simply go along the row headed by "3" (the last row in the table) until we find the identity element "1". Then the number heading the column which contains this "1" is the inverse of 3. We find a "3". Thus 3 is its own inverse. If we seek the inverse of 2, we go along the row headed by "2" (the second last row in the table) until we find the identity element "1". But no "1" appears in this row. Thus 2 has no inverse under multiplication in Z_4.

Let us use the symbol "$^{-}3$" to name the inverse of 3 under addition in Z_4. The symbol is read "the additive inverse of 3" Earlier we said that the inverse of 3 is 1 under addition in Z_4. Thus "$^{-}3$" and "1" are different names for the same number in $(Z_4, +)$. Because of this, another way of writing "$3 + 1 = 0$" would be "$3 + {}^{-}3 = 0$". The following examples show some uses of this new symbol.

Example 1. In $(Z_4, +)$ we have $^{-}2 + 2 = 0$.
To justify this statement examine the definition of "$^{-}2$".

Example 2. In $(Z_4, +)$ we have $^{-}1 + {}^{-}2 = 1$.
To justify this statement we first note that $^{-}1 = 3$ and $^{-}2 = 2$ in $(Z_4, +)$. Why? Convince yourself that "$3 + 2 = 1$" is a true statement in $(Z_4, +)$. If we replace "3" by "$^{-}1$" and if we replace "2" by "$^{-}2$", then we conclude that $^{-}1 + {}^{-}2 = 1$ in $(Z_4, +)$.

Example 3. $^{-}(1 + 2) = {}^{-}1 + {}^{-}2$ in $(Z_4, +)$
The symbol "$^{-}(1 + 2)$" means the additive inverse of $1 + 2$ or, what is the same thing, the additive inverse of 3. We have then

$$^{-}(1 + 2) = {}^{-}3 = 1$$

We saw in Example 2 that $^{-}1 + {}^{-}2 = 1$. Since both $^{-}(1 + 2)$ and $^{-}1 + {}^{-}2$ are equal to 1 we conclude that $^{-}(1 + 2) = {}^{-}1 + {}^{-}2$.

Example 4. "$3 - 1$" and "$3 + {}^{-}1$" name the same number in Z_4. We know that $3 - 1 = 2$. Also $3 + {}^{-}1 = 3 + 3 = 2$. Since both $3 - 1$ and $3 + {}^{-}1$ are equal to 2, we have that $3 - 1 = 3 + {}^{-}1$.

Now that we have a special symbol to represent additive inverses in $(Z_4, +)$, let us select a symbol to represent inverses in $(Z_4, \cdot)$. Since we know that in $(Z_4, \cdot)$ $3 \cdot 1/3 = 1$, let us select the symbol "1/3" to designate the inverse of 3 under multiplication in Z_4. We read this symbol as "1 over 3" or "the multiplicative inverse of 3". Since 3 is its own inverse in $(Z_4, \cdot)$ it is clear that $1/3 = 3$ in $(Z_4, \cdot)$. Similarly $1/1 = 1$ in $(Z_4, \cdot)$. The following examples illustrate some uses of this new symbol.

Example 1. In $(Z_4, \cdot)$ we have $1/3 \cdot 2 = 2$ in $(Z_4, \cdot)$. Note that 1/3, the multiplicative inverse of 3 under multiplication in Z_4, is 3 in $(Z_4, \cdot)$. In short, $1/3 = 3$ in $(Z_4, \cdot)$. Furthermore, $3 \cdot 2 = 2$ is a true statement in $(Z_4, \cdot)$. If we replace "3" in this statement with "1/3", then we conclude that $1/3 \cdot 2 = 2$ in $(Z_4, \cdot)$.

Example 2. The solution set for $3 \cdot 1/x = 1$ is $\{3\}$ in $(Z_4, \cdot)$.
Explain why this is true.

Example 3. The symbol "1/0" does not name any number in Z_4. This is true because 0 does not have a multiplicative inverse in $(Z_4, \cdot)$. Use the table for $(Z_4, \cdot)$ to check that 0 does not have an inverse under multiplication in Z_4.

1.20 Exercises

Note: Unless otherwise stated, all of the exercises in this section should be considered using Z_5 arithmetic.

1. Using your addition table for $(Z_5, +)$ determine the additive inverse of:

a) 2　　d) 3

b) 1　　e) 4

c) 0　　f) $^-2$

2. Using your table for $(Z_5, \cdot)$ determine the multiplicative inverse of:

a) 2　　d) 3

b) 4　　e) 0

c) 1　　f) 1/2

3. Find the simplest names for the following in Z_5:

a) $^-1$　　f) 1/2

b) $^-4$　　g) 1/1

c) $^-0$　　h) 1/4

d) $^-2$　　i) 1/3

e) $^-3$　　j) 1/0

4. Compute the following in $(Z_5, +)$:

a) $3 + {}^-2$　　d) $^-3 + 3$

b) $^-4 + 1$　　d) $^-2 + {}^-4$

c) $^-1 + {}^-3$　　f) $^-0 + 0$

5. Compute the following in $(Z_5, \cdot)$:

a) $1/2 \cdot 4$　　c) $1/3 \cdot 1/2$

b) $1/3 \cdot 3$　　d) $1/2 \cdot 1/4$

6. Solve the following open sentences where the domain of the variable is Z_5.

a) $3 + {}^-3 = x$　　d) $3 \cdot 1/3 = x$

b) $^-4 + 1 = y$　　e) $1/2y = 3$

c) $^-2 + z = 4$　　f) $z \cdot 1/4 = 1/3$

7. Note: the symbol "$^-(^-2)$" means the inverse of the additive inverse of 2 in $(Z_5, +)$.
   a) If we replace the name "$^-2$" in the above symbol with the name "3", then what number in Z_5 do we have?
   b) What number in Z_5 do the following represent?

   1) $^-(^-4)$　　3) $^-(^-3)$

   2) $^-(^-1)$　　4) $^-(^-0)$

   c) What is the additive inverse of the additive inverse of 3?
   d) Can you form a generalization from examining a), b), and c) above?
8. a) Explain why the symbol "1/0" does not name a number in Z_5.
   b) Does "1/0" name a number in W?
   c) Which whole number have inverses in $(W, +)$?
   d) Which whole numbers have inverses in $(W, \cdot)$?
9. The additive inverse of $(3 + 1)$ in $(Z_5, +)$ can be named "$^-(3 + 1)$" or "$^-4$".
   a) What numbers in Z_5 do the following represent?

   1) $^-(1 + 3)$　　4) $^-2 + {}^-4$

   2) $^-1 + {}^-3$　　5) $^-(0 + 2)$

   3) $^-(2 + 4)$　　6) $^-0 + {}^-2$

   b) What is the additive inverse of the sum of 2 and 3 in $(Z_5, +)$?
   c) What is the sum of the additive inverse of 2 and the additive inverse of 3 in $(Z_5, +)$?
   d) Can you form a generalization from examining 9a), b), and c) above?
10. After you have solved 9d) you might experiment with multiplicative inverses in $(Z_5, \cdot)$. Can you find evidence for a corresponding generalization about multiplicative inverses?
11. Two examples of statements which, in general, are true statement about the operational system $(Z_5, +)$ are the following:
    i) If x and y are any elements of Z_5,

    $$x + y = y + x \text{ in } (Z_5, +)$$

    ii) If we let "^-x" mean the additive inverse of x then $x + {}^-x = 0$ in $(Z_5, +)$; where x is any element in Z_5 which has an additive inverse.

    Explain why the following sentences are true, or are not true, for every x and y in Z_5.
    a) In Z_5 arithmetic, the difference $x - y$ is the same as the sum $x + {}^-y$. (To subtract y from x, we may add to x the additive inverse of y. That is, $x - y = x + {}^-y$.)
    b) In Z_5 arithmetic $x - y = {}^-(y - x)$.
    (The additive inverse of $y - x$ is $x - y$.)
    c) $^-(x \cdot y) = {}^-x \cdot {}^-y$

    (The additive inverse of a product is equal to the product of the additive inverses.)

    d) $x \cdot 1/x = 1$ for all x in Z_5.
12. An important property of $(W, \cdot)$ is the following:. Let x and y be any two elements in W. If $x \cdot y = 0$, then $x = 0$ or $y = 0$.
    Explain why there is, or is not, a corresponding property in the following:

    a) $(Z_5, \cdot)$

    b) $(Z_4, \cdot)$

1.21 The Associative and Distributive Properties

The following is a famous grammatical puzzle. "Can you punctuate the string of words in the box so that a correct English sentence is formed?"

John where James had had had had had had had

It turns out we can solve the above puzzle by using the grammatical symbols , . " " Try it!

In mathematics we also make use of symbols which, like grammatical symbols, allow us to write expressions which are clear and correct. The most common grammatical symbols used in mathematics are parentheses. Consider the two expressions given below where addition is to be performed in $(Z_4, +)$.

$$(2 + 3) + 1 \qquad (1)$$

$$2 + (3 + 1) \qquad (2)$$

In (1) we see that "$2 + 3$" has been enclosed in parentheses. The parentheses are used to "signal" that we should consider "$2 + 3$" as naming a single number. Since we are to perform addition in $(Z_4, +)$ this number is 1. Thus, we have

$$(2 + 3) + 1 = 1 + 1 = 2 \qquad \text{in } (Z_4, +)$$

In (2) the parentheses are used to signal that we should consider "$3 + 1$" as naming a single number. Thus, we have

$$2 + (3 + 1) = 2 + 0 = 2 \qquad \text{in } (Z_4, + 1)$$

We note that the result of adding the numbers in (1) was the same as the result of adding the numbers in (2). A question that we might ask is the following: If a, b, and c are any triple of numbers in Z_4, will it always be true that

$$a + (b + c) = (a + b) + c \quad \text{in } (Z_4, +)?$$

If the answer to the question is "Yes", for all triples of numbers in Z_4, then we say that addition is associative in $(Z_4, +)$.

Next let us examine the same triple of numbers in Z_4 but ask if subtraction is associative in $(Z_4, -)$. Consider (3) and (4) below:

$$(2 - 3) - 1 \qquad (3)$$

$$2 - (3 - 1) \qquad (4)$$

We are asking if the result of computing (3) in $(Z_4, -)$ is the same as the result of computing (4) in $(Z_4, -)$. Because of the parentheses in (3) we first compute $2 - 3$ in $(Z_4, -)$. Carrying out the subtractions in (3) we have

$$(2 - 3) - 1 = 3 - 1 = 2 \qquad \text{in } (Z_4, -)$$

However from (4) we have

$$2 - (3 - 1) = 2 - 2 = 0$$

Thus, $\quad 2 - (3 - 1) \neq (2 - 3) - 1 \;$ in $(Z_4, -)$

and we say that subtraction is not associative in $(Z_4, -)$; not associative, because it failed for at least one triple of numbers.

Up to now when we sought out such properties as "commutativity" or "associativity" we confined ourselves to a single operation on a set of numbers. The next property that we shall investigate has a different role to play. It deals with two operations on a set of numbers. Let us consider the following two expressions where addition is to be performed in $(Z_4, +)$ and multiplication is to be performed in $(Z_4, \cdot)$ We shall indicate that we are working with one set and two operations by writing "$(Z_4, +, \cdot)$".

$$2 \cdot (3 + 1) \qquad (5)$$

$$(2 \cdot 3) + (2 \cdot 1) \qquad (6)$$

Again, the parentheses "signal" how our computations should proceed. In (5) since $3 + 1$ is considered as a single number we compute as follows:

$$2 \cdot (3 + 1) = 2 \cdot 0 = 0 \quad \text{in } (Z_4, +, \cdot).$$

We compute (6) as follows:

$$(2 \cdot 3) + (2 \cdot 1) = 2 + 2 = 0 \quad \text{in } (Z_4, + \; \cdot).$$

Since the computation in (5) and (6) both resulted in 0, we conclude that

$$2 \cdot (3 + 1) = (2 \cdot 3) + 2 \cdot 1.$$

If for every triple of numbers a, b, and c in Z_4 it is true that

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{in } (Z_4, +, \cdot),$$

then we say that multiplication is distributive over addition in $(Z_4, +, \cdot)$.

Let us examine the same triple of numbers in Z_4 but ask instead, "Is addition distributive over multiplication in $(Z_4, +, \cdot)$?" Here we must compute the following in $(Z_4, +, \cdot)$:

$$2 + (3 \cdot 1) \qquad (7)$$

$$(2 + 3) \cdot (2 + 1) \qquad (8)$$

In (7) we have $2 + (3 \cdot 1) = 2 + 3 = 1$ in $(Z_4, +, \cdot)$.

In (8) we have $(2 + 3) \cdot (2 + 1) = 1 \cdot 3 = 3$ in $(Z_4, +, \cdot)$.

We conclude that $2 + (3 \cdot 1) \neq (2 + 3) \cdot (2 + 1)$ and that addition is not distributive over multiplication in $(Z_4, +, \cdot)$.

Compute the following in $(Z_4, \cdot, -)$:

$$2 \cdot (3 - 1) \qquad (9)$$

$$2 \cdot 3 - 2 \cdot 1 \qquad (10)$$

From (9) we have $2 \cdot (3 - 1) = 2 \cdot 2 = 0$
From (10) we have $2 \cdot 3 - 2 \cdot 1 = 2 - 2 = 0$
Note: Although $2 \cdot (3 - 1) = 2 \cdot 3 - 2 \cdot 1$ in $(Z_4, \cdot, -)$, We cannot yet conclude that multiplication is distributive over subtraction in $(Z_4, \cdot, -)$. Recall that the property must hold for all triples of numbers in Z_4. Experiment further with other triples of numbers and make a conjecture concerning the existence of a distributive property of $\cdot$ over $-$ in $(Z_4, \cdot, -)$.

1.22 Exercises

1. Compute the following in $(Z_5, +)$:
   - a) $(2 + 4) + 3$
   - b) $2 + (4 + 3)$
   - c) $1 + (2 + 3)$
   - d) $(1 + 2) + 3$
   - e) $(3 + 0) + 4$
   - f) $3 + (0 + 4)$
   - g) $4 + (3 + 3)$
   - h) $(4 + 3) + 3$
2. Compute the following in $(Z_5, \cdot)$:
   - a) $(2 \cdot 4) \cdot 3$
   - b) $2 \cdot (4 \cdot 3)$
   - c) $1 \cdot (2 \cdot 3)$
   - d) $(1 \cdot 2) \cdot 3$
   - e) $(3 \cdot 0) \cdot 4$
   - f) $3 \cdot (0 \cdot 4)$
   - g) $4 \cdot (3 \cdot 3)$
   - (h) $(4 \cdot 3) \cdot 3$
3. a) State a property that is suggested by Exercise 1.
   b) State a property that is suggested by Exercise 2.
   c) Test out these properties on different triples of numbers in Z_5.
4. Compute the following in $(Z_5, +, -)$:
   - a) $2 \cdot (4 + 3)$
   - b) $(2 \cdot 4) + (2 \cdot 3)$
   - c) $1 \cdot (2 + 3)$
   - d) $(1 \cdot 2) + (1 \cdot 3)$
   - e) $3 \cdot (0 + 4)$
   - f) $(3 \cdot 0) + (3 - 4)$
   - g) $4 \cdot (3 + 3)$
   - h) $(4 \cdot 3) + (4 \cdot 3)$
5. State a property suggested by Exercise 4.
6. Compute the following in $(Z_5, +, \cdot)$:
   - a) $^-4 \cdot (3 + {}^-3)$
   - b) $(^-4 \cdot 3) + (^-4 \cdot {}^-3)$
   - c) $^-2 \cdot (5 - {}^-3)$
   - d) $(^-2 \cdot 5) - (^-2 \cdot {}^-3)$
7. a) Is multiplication distributive over subtraction in $(Z_5, \cdot, -)$?
   b) Is division distributive over addition in $(Z_5, \div, -)$?
8. The property $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ is more properly referred to as a "left hand" distributive property of $\cdot$ over $+$. Is there a corresponding "right hand" distributive property namely, $(a + b) \cdot c = a \cdot c + b \cdot c$, in $(Z_5, +, \cdot)$?
9. Assume that for all a, b, and c in Z_5 that

   $a \cdot (b + c) = a \cdot b + a \cdot c$ in $(Z_5, +, \cdot)$.

   a) Using known properties of $(Z_5, +)$ and $(Z_5, \cdot)$ can you prove that
   $a \cdot (b + c) = (c \cdot a) + (b \cdot a)$?

   b) Would $a \cdot (b + c) = (c \cdot a) + (b \cdot a)$ "hold" in $(Z_m, +, \cdot)$ where a, b, and c are any elements of Z_m?

1.23 Summary

1. In this chapter we studied a collection of finite sets called "clock numbers." We found that there were applications of these sets dealing with dials, routes, rotations, codes, etc.
2. We defined addition, subtraction, multiplication, and division on these finite sets.
   (a) We found there were similarities between clock number arithmetic and whole number arithmetic: Both $(W, +)$ and $(Z_m, +)$ are operational systems. There are corresponding properties for both arithmetics dealing with an identity element for multiplication, an identity element for addition, commutative properties for addition and multiplication, associative properties for both addition and multiplication, a distributive property of multiplication over addition. Division by the additive identity is not defined in either arithmetic.
   (b) We found there were differences between clock number arithmetic and whole number arithmetic: The sets Z_m are finite whereas the set W is not finite. Subtraction is not an operation on the set W whereas $(Z_m, -)$ is an operational system. Every element in Z has an additive inverse whereas only 0 in W had an additive inverse.
   The solution sets for corresponding open sentences in $(W, \cdot)$ and $(Z_m, \cdot)$ can differ greatly. For example: The open sentence $3 \cdot x = 3$ has the solution set $\{1\}$ in $(W, \cdot)$ whereas the corresponding open sentence in $(Z_6, \cdot)$ has the solution set $\{1, 3, 5\}$.
   (c) We found there were many similarities between the various $(Z_m, +)$ operational systems that we examined. In your future study more will be learned about the operational systems $(Z_m, +, \cdot)$.
3. New terms were introduced and used. Among these were "statement", "variable", "open sentence", "is less than", "is greater than", "equation", "inequation", "solution set", "empty set", "additive inverse", "multiplicative inverse", "associativity", "distributivity". Check over the above terms to see if you understand what they mean. Where there is doubt recheck the meaning s given in the text.

4. As you continue to study mathematics many of the ideas and terms found in this chapter will be given precise definitions and meanings. In particular, Chapter 2 will explore the idea of "operational system" by considering many new and interesting operations on sets.

1.24 Review Questions

Make up tables for $(Z_8, +)$ and $(Z_8, \cdot)$.

1. Compute the following in $(Z_8, +)$:

a) $6 + 7$ d) $7 + (7 + 6)$

b) $5 + 3$ e) $3 + (7 + {}^-5)$

c) $(7 + 7) + 6$ f) ${}^-3 + ({}^-5 + {}^-3)$

2. Compute the following in $(Z_8, \cdot)$:

a) $6 \cdot 7$ d) $(3 \cdot 4) \cdot 5$

b) $2 \cdot 4$ e) $1/3 \cdot (1/5 \cdot 1/7)$

c) $3 \cdot (4 \cdot 5)$ f) $1/5 \cdot (3 \cdot 7)$

3. Compute the following in $(Z_8, +, \cdot)$:

a) $3 \cdot (7 + 5)$ c) $6 \cdot (7 - 5)$

b) $(3 \cdot 7) + (3 \cdot 5)$ d) $6 \cdot (7 + {}^-5)$

4. Is $(Z_8, \cdot)$ an operational system? Explain.

5. Let a and b be any elements of Z_8. Explain why, or why not, the following is true in $(Z_8, \cdot)$.

If $a \cdot b = 0$, then $a = 0$ or $b = 0$

6. List all the elements of Z_8 and their corresponding inverses in $(Z_8, +)$.

7. List all the elements of Z_8 and their corresponding inverses in $(Z_8, \cdot)$.

8. Solve the following open sentences where the domain of the variable is Z_8.

a) $3 + x = 5$ f) $4 \cdot z = 0$

b) $y + 2 = 6$ g) $3 \cdot x = 7$

c) $3 \cdot x = 5$ h) ${}^-5 \cdot x = 7$

d) $2 \cdot y = 0$ i) $2 \cdot y = 3$

e) $3 - 7 = x$ j) $4 \cdot (3 + 5) = x$

9. If today is Sunday, then what day of the week is 1000 days from today? Explain your answer.

10. A circular bus route has 20 stops each 5 minutes apart. Which "stop" should the relief bus driver go to after the bus has been out seven and one quarter hours? Call the place where the route begins "stop 0," and call the first stop after this "stop 1," etc. .

CHAPTER 2: Sets and Operations

2.1 Ordered Pairs of Numbers and Assignments

Suppose you were given a pair of whole numbers, say 6 and 2, and were asked to assign a third number to this pair. Such an instruction might seem unclear, and indeed there are an endless number of answers that could be given. For example, one person might assign the number 8, since 6 + 2 = 8. We could show this assignment simply by writing

$$(6, 2) \longrightarrow 8$$

to indicate that the pair of numbers (6, 2) yields the number 8 if one is thinking of addition.

Another person, given the pair of numbers (6, 2), might write

$$(6, 2) \longrightarrow 3$$

and say that the pair (6, 2) yields the number 3. We would probably guess that such a person is thinking of division, and we could write "6 ÷ 2 = 3".

If we were given the pair (2, 6) and thought of division, we would write

$$(2, 6) \longrightarrow 1/3$$

Since 2 ÷ 6 = 1/3. Thus, the pair (2, 6) does not produce the same number as the pair (6, 2). The order of the numbers in the pair is important. For this reason, we often speak of an ordered pair of numbers. In the ordered pair (6, 2), 6 is the first component of the pair and 2 the second.

Question: Are the ordered pairs (6, 2) and (2, 6) assigned the same number if one is thinking of addition?

Below are several ordered pairs of numbers. Each pair has been assigned a third number. In each case, tell how you think the third number was assigned.

$$(3, 2) \longrightarrow 5$$
$$(3, 2) \longrightarrow 1$$
$$(2, 3) \longrightarrow 5$$
$$(3, 2) \longrightarrow 6$$
$$(2, 3) \longrightarrow 6$$
$$(3, 2) \longrightarrow 9$$
$$(2, 3) \longrightarrow 8$$

The last two assignments in the above list result from raising a number to a power. Given the ordered pair (3, 2), raising 3 to the power 2 means that we are to use 3 as a factor twice – that is, 3 × 3 – obtaining 9. This is often written as

$3^2 = 9$ (9 is a power of 3: specifically, 9 is the second power of 3.)

Similarly, given the ordered pair (2,3), we may think of 2 raised to the power 3. This means that we are to use 2 as a factor 3 times. Thus,

$2^3 = 2 \times 2 \times 2 = 8$. (8 is the third power of 2.)

This explains the assignments $(3, 2) \longrightarrow 9$ and $(2, 3) \longrightarrow 8$, Clearly, if one is thinking of raising a number to a power, the ordered pairs (3, 2) and (2,3) are not assigned the same number.

Questions: What number is assigned to the ordered pair (2, 2) by the process of raising to a power? To (3, 3)? (Notice that the same number may be used for both the first and second component of an ordered pair.)

It is often convenient to use a table (as we did in Chapter 1) to show numbers assigned to pairs of numbers. For example, at the left below is a table showing some of the assignments made if one thinks of addition. At the right is a table showing some assignments if one is thinking of raising to a power.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

	1	2	3	4	5
1	1	1	1	1	1
2	2	4	⑧	16	32
3	3	⑨	27	81	243
4	4	16	64	256	1024
5	5	25	125	625	3125

Do you see how the entries in the tables were obtained? Notice that in the second table the entries "9" and "8" have been circled, emphasizing that (3, 2) and (2, 3) yield different results.

Question: Suppose a is some whole number. What number is assigned to the ordered pair (a, 1) by the process of raising to a power? How is this shown in the table of powers above?

Below is still another table showing assignments of numbers to pairs of numbers. These assignments should be familiar from your work in Chapter 1; do you see how they were obtained?

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2.2 Exercises

1. Tell what number is assigned to the following ordered pairs by usual addition.
   (a) (5,0) (b) (0,5) (c) (6,6) (d) (218,365)
   (e) (365,218) (f) (750,250) (g) (2/3, 1/5)
   (h) (4-1/2, 2-3/4) (i) (.83, .27)
   (j) (2000000, 8000000).
2. a. Working with whole numbers only list all ordered pairs of whole numbers to which the number 5 is assigned by addition. (Remember that (a,b) and (b,a) are different pairs)
   b. Again using whole numbers only, list all ordered pairs to which the number 1 is assigned by addition.
   c. List all ordered pairs of whole numbers to which the number 0 is assigned by addition.
3. a. List all ordered pairs of whole numbers to which 24 is assigned by multiplication.
   b. List all ordered pairs of whole numbers to which the number 13 is assigned by multiplication.
   c. List all ordered pairs of whole numbers to which 0 is assigned by multiplication.
4. Tell what number is assigned to the following ordered pairs by multiplication.

   (a) (5,0) (b) (0,5) (c) (8,6) (d) (51,106)

   (e) (106,51) (f) (4-1/2, 2-3/4)
5. In the text, we explained raising a whole number to a power. 4^2means 4×4 or 4 used as a factor 2 times. Often in mathematics, we use a raised dot "·" instead of an "×" to indicate multiplication. Thus, we may write $4^2 = 4 \cdot 4 = 16$. An expression such as "a^b" is read as "a to the b power" and the number b is called an exponent. We are assuming that both a and b are whole numbers. With this in mind, tell what numbers the following name.

   (a) 2^3 (b) 2^2 (c) 2^1 (d) 10^1 (e) 10^2 (f) 10^3

   (g) 10^4 (h) 10^6 (i) 5^2 (j) 2^5 (k) 4^3 (l) 3^4

   (m) 3^3 (n) 1^{546}
6. If we think of "raising to a power" as assigning numbers to ordered pairs of numbers, what number is assigned to the following pairs by "raising to a power"? Remember that we take the second number as the exponent.

   (a) (3,4) (b) (4,3) (c) (4,2) (d) 2,4) (e) (3,5)

   (f) (5,3)
7. a. List all ordered pairs of whole numbers which are assigned the number 16 under "raising to a power."
   b. List all ordered pairs of whole numbers which are assigned the number 10 by "raising to a power.".
8. We know that assignments of numbers to pairs can be shown by a table. Fill in all the cells in the following table for addition. (Notice in this case that the numbers are not listed in any particular order.)

+	5	682	17	8	0	1	1720
5							
682							
17							
8							
0							
1							
1720							

9. In this problem, we look at another way of assigning a number to an ordered pair of numbers. Consider the ordered pair (24, 16) of whole numbers. The set of whole numbers which divide 24 is

   $$\{1, 2, 3, 4, 6, 8, 12, 24\}$$

   the set of
   whole numbers which divide 16 is

   $$\{1, 2, 4, 8, 16\}$$

   Notice there are some numbers (1, 2, 4, and 8) which divide both 24 and 16. Of these, 8 is the greatest. Therefore, we call 8 the greatest common divisor of 24 and 16. If we agree to assign the greatest common divisor to the ordered pair (24, 16), we will write

   $$(24, 16) \longrightarrow 8$$

   Under this same scheme, we would make the assignment

   $$(12, 18) \longrightarrow 6$$ (Do you see why?)

   Use the "greatest common divisor" idea to make assignments to the following ordered pairs:

   (a) (6,8) (b) (6,12) (c) (10,15) (d) (100,200)

   (e) (21,45) (f) (7,9) (g) (1,10) (h) (4,4)

   (i) (21,42) (j) (42,21)
10. a. In what cases do the ordered pairs of whole numbers (a,b) and (b,a) produce the same number under addition?
    b. In what cases do the ordered pairs of whole

numbers (a,b) and (b,a) produce the same number under multiplication?

c. In what cases do the ordered pairs of whole numbers (a,b) and (b,a) produce the same number under "raising to a power"?

11. In the list of expressions below, n represents a whole number:

$$3n^2$$

$$3n^2 + 2$$

$$3n^2 - 2$$

$$1/2n^2$$

$$1/2n^2 + 8$$

$$2n^3 + 5$$

(a) Find what whole number each expression if n = 0. (Warning: in some cases, there may be no whole number.)

(b) Find what whole number each expression represents if n = 2.

(c) Find what whole number each expression represents if n = 5.

(d) Find what whole number each expression represents if n = 10.

(e) Find what whole number each expression represents if n = 100.

2.3 What Is An Operation?

You know from arithmetic that given an ordered pair of whole numbers, we can assign a number called their sum. For example,

$$(3, 5) \longrightarrow 8$$

Because addition assigns to every ordered pair one and only one whole number which is their sum, we have called (in Chapter 0) addition an operation on W. We referred to the ordered pair (W, +) as an operational system.

There are many interesting operations on the set of whole numbers. As an example, consider the "maximizing" operation. To illustrate the way the maximizing operation assigns whole numbers to ordered pairs of whole numbers, consider the ordered pair (6, 2). Of the two numbers making up the pair, 6 is the larger, therefore, we assign 6 to the pair.

$$(6, 2) \longrightarrow 6$$

As another illustration, under this operation we assign 10 to the ordered pair (3, 10). To every ordered pair (a,b), we assign the larger of the two numbers, a and b. It is possible that a and b may be the same number, as in the pair (3, 3). In such a case, we shall simply assign the number itself to the pair:

$$(3, 3) \longrightarrow 3$$

Do you see that here again there is no doubt about the number to be assigned? Every ordered pair of whole numbers is assigned one and only one whole number. Therefore, like addition, maximizing is an operation on the set of whole numbers.

Now suppose we consider "taking the average" of two whole numbers. (The average we are speaking of here is more properly called the arithmetic mean.) The average of 5 and 13 is 9, the average of 6 and 10 is 8. If we write these as assignments, we have the following:

$$(6, 10) \longrightarrow 8;$$

$$(5, 13) \longrightarrow 9.$$

Now, however, take a pair such as (5, 8). There is no whole number which is the average of 5 and 8. You may know that 6-1/2 is the average here, but 6-1/2 is not a whole number. If we are working only with whole numbers, there is no number to be assigned to the pair (5, 8). Therefore, averaging is not an operation on the whole numbers because we have a pair of whole numbers to which no assignment can be made.

Let us look at each of these examples again:

Addition: Here we have $(3, 5) \longrightarrow 8$. Since we have the well known symbol "+" for addition, we could just as well write

$$(3, 5) \longrightarrow 3 + 5.$$

In this case it is easier to write "8", but suppose we want to talk about any pair of whole numbers. We might designate this pair as "'a, b)" and then write

$$(a, b) \longrightarrow a + b, \quad \text{for every whole number a and every whole number b}$$

under the operation of addition.

Maximizing: In working with this operation over the whole numbers, we can write, $(8, 3) \longrightarrow 8$. And if we use "max (a, b)" to mean the greater of the two numbers a and b, we can write in general

$$(a, b) \longrightarrow \text{Max}(a, b), \quad \text{for every whole number a and every whole number b}$$

under the operation of maximizing. In the case of addition, the symbol "+" is written between the symbols "a" and "b". We can also do this in the case of "max" and write

$$(a, b) \longrightarrow a \max b, \quad \text{for every whole number a and every whole number b}$$

Averaging: We do not have a symbol for the average of two numbers. But again we can invent one. Let us agree that "aVb" shall mean, "the average of the whole numbers a and b." Thus, "6V8 = 7" is just another way of indicating the assignment

$$(6, 8) \longrightarrow 7$$

if one is thinking of averaging. As we saw earlier, however, 5V8 is not a whole number, and V is not an operation over the whole numbers

Question: Name five other ordered pairs (a, b) of whole numbers for which aVb is not a whole number.

Below are three tables showing the assignments for certain pairs of whole numbers under the "+" and "max" operations and for averaging. An important point to notice is that there are open cells in the table for V (averaging). The fact that these cells are open emphasizes once again why averaging is not an operation on the whole numbers; there just are not any whole numbers which properly go in these cells.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

max	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
2	2	2	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

V	0	1	2	3	4
0	0		1		2
1		1		2	
2	1		2		3
3		2		3	
4	2		3		4

We have seen three symbols – "+", "max", and "V" – used to denote schemes for assigning numbers to ordered pairs of whole numbers. V is not an operation since aVb is not a whole number for every ordered pair (a, b) of whole numbers. On the other hand, + and max are operations since a + b and a max b are, for every pair (a, b), unique whole numbers. These examples lead us to a general definition of an operation on the set of whole numbers.

Definition: Let * be a scheme for assigning numbers to ordered pairs of whole numbers.
If * assigns to each ordered pair (a, b) of whole numbers one and only one whole number, then * is a binary operation on the set of whole numbers.

The word "binary" in this definition is worth some attention. The prefix "bi-" is associated with the idea of a pair, or two things (think of "bicycle" and "biped", for instance). Thus, a binary operation is one which assigns a number to a pair of numbers. Suppose c is the whole number assigned to the ordered pair (a, b) by operation *. Then we write

$$a * b = c.$$

(If you think again about addition of whole numbers, you will see that this is what we have always done there. For instance, + assigns the number 10 to the ordered pair (6, 4), and we write " 6 + 4 = 10",)

The notion of operation is a very general one and may be applied to any set, not just the set W of whole numbers. As one example, consider again the operation of maximizing, This time we shall work with the set $S = \{1, 2, 3, 4\}$, which is a finite subset of the whole numbers. The table below shows a max b for the ordered pairs (a, b) of numbers in S.

max	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

Notice from this table that for every ordered pair (a, b) of numbers in S, a max b is a number in S. Therefore, "max" is an operation on the set S as well as on the set W of whole numbers.

The last example suggests a more general definition of operation.

Definition: A binary operation * on a set S is an assignment which assigns to each ordered pair (a, b) of elements in S, one and only one element c in S.

This definition says essentially the same thing as the earlier one, except that this time we did not restrict ourselves to the set W of whole numbers. In fact, the elements of S need not be "numbers" at all! (See exercise 16.) We denote the operational system consisting of the set S and the operation * by "(S, *)".

2.4 Exercises.

1. What number does each of the following ordered pairs of whole numbers produce under the operation of maximizing discussed in the text? (When "a" is used, it is intended that a is a whole number.)

   (a) (0,0) (b) (0,1) (c) (1,0) (d) (5,15) (e) (15,5)

   (f) (30,100) (g) (2010,2008) (h) (999,1000)
   (i) (a,a+1) (k) (a,1·a) (l) (a,0).

2. Evaluate each of the following:

   (a) 6 max 2 (b) 6 + 2 (c) 6· 2 (d) 6 – 2 (e) 6 ÷ 2

   (f) 588 + 92 (g) 1001 – 865 (h) 88 × 97 (i) 483 ÷ 3

(j) 82 × 10,000 (k) 4.3 × 100

3. Is subtraction an operation on the set of whole numbers?
(Hint: Does subtraction assign a whole number to the ordered pair (2, 5)?)
4. Is division an operation on the set of whole numbers?
5. Suppose we decide to assign to every ordered pair of whole numbers (a,b) every whole number which divides both a and b. Is such a scheme an operation on the whole numbers? Why or why not?
6. In this problem, let "a * b" mean "the greatest common divisor of a and b." (See problem 9 of Section 2.2)
   (a) Is * an operation on the set

   $$W = \{0, 1, 2, 3, 4, \ldots\}?$$

   (b) Is * an operation on the set

   $$N = \{1, 2, 3, 4, 5, \ldots\}?$$

7. In this problem, we shall consider a new way of assigning a number to an ordered pair of numbers, specifically an ordered pair of natural numbers. (the whole numbers except 0) To explain it, we shall use the ordered pair (6, 8). Now 1 × 6 = 6, 2 × 6 = 12, 3 × 6 = 18, etc. Therefore, 6, 12, 18 etc. are called multiples of 6. The list of multiplies of 6 may be indicated as follows:

   6, 12, 18, 24, 30, 36, 42, 48, 54, . . .

   In the same way, the multiples of 8 may be shown in the following way:

   8, 16, 24, 32, 40, 48, 56, 64, . . .

   Of course, 6 and 8 have some multiples in common, such as 24, 48, 96, etc. Of these, 24 is the smallest number and we shall call it the least common multiple of 6 and 8. In this problem, let us agree that

   lcm (a, b)

   means "the least common multiple of a and b;" for example, lcm (8,6) = 24. Also, lcm (10, you 15) = 30; do you see why?
   Evaluate the following:
   (a) lcm (2, 3)
   (b) lcm (5, 10)
   (c) lcm (10, 5)
   (d) lcm (7, 11)
   (e) lcm (11, 7)
   (f) lcm (1, 5)
   (g) lcm (5, 1)
   (h) lcm (100,000)
   (i) lcm (90,70)
   (j) lcm (14, 42)
8. Suppose we continue with the notion of least common multiple, used in problem 7. But this time let us work with the set W of whole numbers, which means that 0 is now included in our set. Thus, the set of multiples of 6 is

   {0, 6, 12, 18, 24, 30, 36, . . . }

   Zero is included since 0 × 6 = 0. Similarly, the set of multiples of 8 is

   {0, 8, 16, 24, 32, 40, 48, . . . }

   Now, with the understanding that "lcm (a, b)," means "least common multiple of a and b," where a and b are whole numbers, evaluate the following:
   (a) lcm (2, 3)
   (b) lcm (5, 10)
   (c) lcm (10, 5)
   (d) lcm (7, 11)
   (e) lcm (11,7)
   (f) lcm (1, 5)
   (g) lcm (5, 1)
   (h) lcm (100, 100)
   (i) lcm (90, 70)
   (j) lcm (14, 42)
   (k) lcm (9, 9)
9. Answer the following questions on the basis of your work in problems 7 and 8:
   (a) Is lcm an operation on the set N of natural numbers?
   (b) Is lcm an operation on the set W of whole numbers?
   Be prepared to defend your answers.
10. Consider the set S = {0, 1} which is a finite set containing exactly two numbers.
   (a) Is ordinary addition an operation on set S? Construct a table showing all possible products.
   (b) Is ordinary multiplication an operation on set S? Construct a table showing all possible products.
11. The set of even whole numbers is indicated below:

   {0, 2, 4, 6, 8, 10, . . . }.

   (a) Is addition an operation on the set of even whole numbers?
   (b) Is multiplication an operation on the set of even whole numbers?
   (c) Is raising to a power an operation on the set of even whole numbers?
12. The set of odd whole numbers is indicated below:

   {1, 3, 5, 7, 9, 11, . . . }.

   (a) Is addition an operation on the set of odd whole numbers?

(c) Is raising to a power an operation on the set of odd whole numbers?

13. In Chapter 1, we worked with some finite systems tems. In this problem, we shall use the system $(Z_2, +)$. (A physical model for this system is furnished by a clock face with numerals "0" and "1".)
    (a) Construct a table for $(Z_2, +)$.
    (b) According to the definition of operation in Section 2.3, is + an operation on the set Z_2? Why or why not?

14. Let S be a set that has two elements, a and b. That is, S = {a, b}. We don't know what "things" a and b are, but suppose we are told that a is assigned to the ordered pair (a, a), b is assigned to the ordered pair (a, b) b is assigned to the ordered pair (b, a), and a is assigned to the ordered pair (b, b). These assignments are displayed in the table below:

	a	b
a	a	b
b	b	a

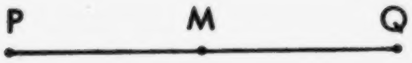
Does this table define an operation on the set {a, b}? Compare the table to that in part (a) of problem 13. Do you see any similarities?

15. Consider the following table:

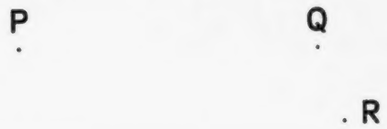
	a	b
a	a	b
b	b	c

   (a) Does this table define an operation on the set {a, b}?
   (b) Does the table define an operation on the set {a, b, c}?
   (c) Does the table define an operation on the set {a, b, c, d}?

16. Although we have not yet talked about geometry in this course, you probably have some idea of what a point is. Given two points, you can find the point midway between them; this point may be called the midpoint of the two given points. For example, P and Q are two points in the drawing below, and M is the midpoint.



   (a) Given any two points, is there one and only one midpoint?
   (b) Suppose you were given the pair of points (Q, Q). (Thus, the "two" points are really the same point.) What point would you assign as midpoint?
   (c) Consider the set of all ordered pairs of points. If mid (P, Q) means "the midpoint of P and Q", is mid an operation on the set of pairs of points?
   (d) If P, Q, and R are three points as below, what is mid (mid (P, Q), R)?



   Is mid (mid(P,Q),R) the same point as mid(P,mid(Q,R))?

2.5 Computations with Operations.

You know by now what is meant by a binary operation on a set. You have seen that the symbol "*" is often used for an operation (we have used special symbols such as "+" and "max" also). In fact, any symbol at all may be used for a particular operation, as long as it is clear to what operation the symbol refers. In this and following sections, we are going to work with several different operations, and it would be troublesome to invent a new symbol for each one of them. On the other hand, we cannot use "*" for all of them. Therefore, we shall make use of subscripts, and denote the operations by symbols such as

$$*_1, *_2, *_3, \text{ etc.}$$

Now let us define six different operations, some of them familiar and others probably new to you.

$*_1$	$a *_1 b = a \cdot b$	In other words, the $*_1$ operation is simply ordinary multiplication of whole numbers. For example, $5 *_1 3 = 15$.
$*_2$	$a *_2 b = a + b$	For example, $5 *_2 3 = 8$.
$*_3$	$a *_3 b = a \max b$	For example, $5 *_3 3 = 5$.
$*_4$	$a *_4 b = a$	In other words, this operation assigns to every ordered pair the first number of the pair.

		For example, $5 *_4 3 = 5$. But $3 *_4 5 = 3$.
$*_5$	$a *_5 b = 17$	Notice that this operation assigns the same number to every pair!
$*_6$	$a *_6 b = a^2 + b^2$	For example, $5 *_6 3 = 5^2 + 3^2 = 25 + 9 = 34$.

In order to see how to compute with these six operations, we look at some examples.

Example 1. Find $3 *_6 2$

The $*_6$ operation assigns to every ordered pair (a, b) the number $a^2 + b^2$. In our example, a is 3 and b is 2. Therefore,

$$3 *_6 2 = 3^2 + 2^2 = 9 + 4 = 13.$$

Example 2. Find $(3 *_6 2) *_2 4$.

The fact that "$3 *_6 2$" has been enclosed in parentheses means that we are to consider this as a single number. And, from Example 1, we know that this number is 13. Hence, we may write

$$(3 *_6 2) *_2 4 = 13 *_2 4$$

But the $*_2$ operation is ordinary addition of whole numbers; so $13 *_2 4 = 17$. Therefor , we have

$$\begin{aligned}(3 *_6 2) *_2 4 &= 13 *_2 4\\ &= 17.\end{aligned}$$

Example 3. Find $3 *_6 (2 *_2 4)$.

Compare this with Example 2. Although the same numbers and the same operations involved, the parentheses have been differently placed. In this example, we are to consider "$2 *_2 4$" as a single number; since $*_2$ is ordinary addition, this number is 6. Thus we have

$$\begin{aligned}3 *_6 (2 *_2 4) &= 3 *_6 6\\ &= 3^2 + 6^2\\ &= 9 + 36\\ &= 45\end{aligned}$$

(Remember how the $*_6$ operation is defined.)

We see that the results in Examples 2 and 3 are not the same. This points up the importance of parentheses in mathematical expressions.

Example 4. Find $((4 *_4 7) *_6 2) *_1 10$.

This expression contains two different "signals" in the form of parentheses. First, note that $(4 *_4 7)$ is to be taken as a single number. And from the way in which the $*_4$ operation is defined, we know that $4 *_4 7$ is 4, since 4 is the first number of the pair (4, 7).

So, we have for a first step:

$$((4 *_4 7) *_6 2) *_1 10 = (4 *_6 2) *_1 10$$

Do you see that "$4 *_4 7$" has been replaced by "4"?

Next we are to consider $4 *_6 2$ as a single number. This number is 20; do you see why? So we have

$$\begin{aligned}((4 *_4 7) *_6 2) *_1 10 &= (4 *_6 2) *_1 10\\ &= 20 *_1 10\end{aligned}$$

"$4 *_6 2$" has been replaced by "20". Finally, we know that $20 *_1 10$ is 200; Therefore, if all the steps are written together, we have the following:

$$\begin{aligned}((4 *_4 7) *_6 2) *_1 10 &= (4 *_6 2) *_1 10\\ &= 20 *_1 10\\ &= 200.\end{aligned}$$

Sometimes, but not always, when an expression involves more than one pair of parentheses, a pair of brackets may replace a pair of parentheses. For instance, the expression of Example 4 might be written

$$[(4 *_4 7) *_6 2] *_1 10.$$

In the following example, the steps have been listed without any additional explanation. Be sure that you can explain each step.

Example 5. Find $(4 *_2 7) *_4 ((3 *_1 2) *_3 5)$.

$$\begin{aligned}(4 *_2 7) *_4 ((3 *_1 2) *_3 5) &= 11 *_4 (6 *_3 5)\\ &= 11 *_4 6\\ &= 11.\end{aligned}$$

2.6 Exercises.

In problems 1 through 20, the operations are those defined in Section 2.5 of the text.

1. (a) $5 *_1 2 =$ (e) $5 *_5 2 =$

   (b) $5 *_2 2 =$ (f) $5 *_6 2 =$

   (c) $5 *_3 2 =$

   (d) $5 *_4 2 =$

2. (a) $(7 *_3 3) *_3 8 =$

   (b) $7 *_3 (2 *_3 8) =$

(c) $2 *_6 (3 *_6 5) =$

(d) $(2 *_6 3) *_6 5 =$

3. (a) $109 *_3 111 =$

(b) $111 *_3 109 =$

(c) $109 *_4 111 =$

(d) $111 *_4 109 =$

4. (a) $58 *_4 32 =$

(b) $32 *_4 58 =$

(c) $58 *_5 32 =$

(d) $32 *_5 58 =$

5. (a) $42 *_1 1 =$ (d) $615 *_1 1 =$

(b) $42 *_2 0 =$ (e) $615 *_2 0 =$

(c) $42 *_3 0 =$ (f) $615 *_3 0 =$

6. (a) $(3 *_2 5) *_2 4 =$

(b) $3 *_2 (5 *_2 4) =$

(c) $3 *_6 (1 *_6 4) =$

(d) $(3 *_6 1) *_6 4 =$

7. (a) $(7 *_3 5) *_2 8 =$

(b) $7 *_3 (5 *_2 8) =$

8. (a) $5 *_6 (2 *_1 3) =$

(b) $(5 *_6 2) *_1 3 =$

9. (a) $(420 *_5 3) *_1 85 =$

(b) $420 *_5 (3 *_1 85) =$

10. $((14 *_5 3) *_4 2) *_1 10 =$

11. $15 *_3 ((3 *_2 5) *_2 889) =$

12. $([(8 *_3 10) *_5 15] *_1 87 =$

13. $((2 *_6 3) *_6 4) *_6 5 =$

14. $((2 *_2 3) *_2 4) *_2 5 =$

15. $3 *_1 (5 *_2 6) =$

16. $3 *_2 (5 *_1 6) =$

17. $[5 *_1 (2 *_2 3)] *_2 [5 *_2 (2 *_1 3)] =$

18. $(8 *_4 12) *_3 (8 *_5 12) =$

19. $((((2 *_1 2) *_2 2) *_3 2) *_4 2) *_5 2 =$

20. $(((((2 *_1 2) *_2 2) *_3 2) *_4 2) *_5 2) *_6 2 =$

21. Parentheses are also important in expressions involving the ordinary computations in arithmetic, as the following problems illustrate.

(a) $1/2 \times (2/3 + 3/4) =$

(b) $(1/2 \times 2/3) + (1/2 \times 3/4) =$

(c) $[(6\text{-}1/2 \times 3) + 3] \times 10 =$

(d) $(4\text{-}1/2 + 2\text{-}2/3) \div 1/2 =$

(e) $4\text{-}1/2 + (2\text{-}2/3 \div 1/2 =$

(f) $1/2 \div (2 \div 1/3) =$

(g) $(1/2 \div 2) \div 1/3 =$

(h) $(((2\text{-}1/3 \times 1\text{-}1/2) \div 1/2) + 3/4) - 1/2 =$

22. Make up an operation over the whole numbers, and call it $*_7$.
(Caution: Be sure that it is an operation!)
Then compute the following:

(a) $8 *_7 15 =$

(b) $15 *_7 8 =$

(c) $5 *_7 (2 *_7 3) =$

(d) $(5 *_7 2) *_7 3 =$

23. Consider the following expressions:

$$2(n^3); \quad (2n)^3$$

They are not the same. The first one is often written as "$2n^3$", without parentheses

(a) Write a flow chart showing how to compute $2n^3$.

(b) Write a flow chart showing how to compute $(2n)^3$.

(c) Are there any values of n for which

$$2n^3 = (2n)^3?$$

2.7 Open Sentences.

Consider an open sentence such as

$$5 *_2 x = 8$$

The operation $*_2$, according to our definition, is just ordinary addition of whole numbers. Therefore, the question posed (and it is an easy one) is this:

> Is there an ordered pair (5,x) to which 8 is assigned by the operation of addition?

The answer is obvious; x must be 3 in order for this assignment to be made. Therefore, we say that 3 is a solution of the open sentence "$5 *_2 x = 8$." In this case, it is easy to see that 3 is the only solution. But some open sentences have more than one solution; so you must be careful when "solving" an open sentence that you indicate all of the solutions, not just some of them.

Example 1. Solve $3 *_3 n = 3$.

The $*_3$ operation assigns max(a,b) to

every ordered pair (a,b). Therefore, the open sentence will be true if and only if max(3,n) = 3. But this in turn will be true if n is 0, if n is 1, if n is 2. It will also be true if n is 3, since max (max(3,3) = 3. Do you see, however, that it will no longer be true if n is a whole number greater than 3? Therefore, the solution set of the open sentence is:

$$\{0, 1, 2, 3\}$$

In this case, we have exactly four solutions.

Example 2: Solve $2 *_3 a = a$.

Under the $*_3$ operation, the assignment $(2,a) \longrightarrow a$ means that a is the greater of the two numbers, 2 and a. Therefore, in order to make the statement true, a must be 2 or any number greater than 2. There are infinitely many solution!
The solution set is

$$\{2, 3, 4, 5, \ldots\}$$

Example 3. Solve $a *_6 2 = 29$.

From the definition of the operation $*_6$ we know that if this sentence is to be true, then $a^2 + 2^2$ must be 29. But $2^2 = 4$; so, $a^2 + 4$ must be 29. Now, if $a^2 + 4$ must be 29, do you see that a^2 must be 25?
However, 25 is not a solution; we are looking for a, not a^2. But of course if a^2 is 25, then a is 5.
The complete list of steps might be written as follows:

$$a^2 + 4 = 29$$
$$a^2 = 25$$
$$a = 5.$$

Is 5 the only whole number solution?

2.8 Exercises

In 1-26 find the solutions of the open sentences, using the indicated operations as defined in Section 2.5.
If there is no solution, say so. If there is more than one solution, be sure to find all of them.
Use only whole numbers.

1. $8 *_2 a = 11$
2. $11 *_2 a = 8$
3. $5 *_1 a = 10$
4. $10 *_1 a = 5$
5. $n *_2 81 = 103$
6. $n *_2 103 = 81$
7. $n *_1 17 = 187$
8. $n *_1 187 = 17$
9. $5 *_3 a = 5$
10. $a *_3 6 = 6$
11. $42 *_4 a = 21$
12. $a *_4 42 = 42$
13. $42 *_4 a = 42$
14. $85 *_5 a = 17$
15. $85 *_5 a = 18$
16. $2 *_6 a = 13$
17. $a *_6 2 = 13$
18. $3 *_6 a = 25$
19. $3 *_6 a = 30$
20. $5 *_4 a = 10$
21. $n *_4 15 = 60$
22. $3 *_5 n = 3$
23. $52 *_3 n = 1$
24. $32 *_1 * = 321$
25. $n *_2 32 = 321$
26. (a) $832 *_1 a = 832$
    (b) $832 *_2 a = 832$
    (c) $832 *_3 a = 832$
    (d) $832 *_4 a = 832$
    (e) $832 *_5 a = 832$
    (f) $832 *_6 a = 832$
27. Before solving the following open sentences, it is important to understand the following:
    Suppose you are asked to solve the open sentence "$a + a = 6$," where "+" is ordinary addition. Since $3 + 3 = 6$, 3 is a solution. Notice that "a" is used more than once in the sentence, and the same number must be used for each "a" in the sentence. Thus, although $4 + 2 = 6$, this does not give us a solution to the sentence.
    (a) $3 *_1 a = a *_1 3$
    (b) $3 *_2 a = a *_2 3$
    (c) $3 *_3 n = n *_3 3$
    (d) $3 *_4 n = n *_4 3$

(e) $3 *_5 x = x *_5 3$

(f) $3 *_6 x = x *_6 3$

28. (a) $a *_6 a = 72$
(b) $a *_1 a = 25$
(c) $n *_1 n = 24$
(d) $n *_2 n = 242$
(e) $a *_2 a = 243$
(f) $a *_1 a = 2$
(g) $x *_6 x = 17$
(h) $x *_3 x = 5$
(i) $a *_4 a = 5$
(j) $a *_5 a = 5$

29. (a) $n *_1 (n *_1 n) = 8$
(b) $a *_2 (a *_2 a) = 9$
(c) $a *_5 (a *_5 a) = 17$
(d) $n *_3 (n *_3 n) = 23$
(e) $(a *_6 a) *_6 a = 68$
(f) $n *_4 (n *_4 n) = 108$

2.9 *Properties of Operations*

Referring to the operations defined in Section 2.5, tell what number each of the following expressions names:

$5 *_3 2$	$5 *_4 2$
$2 *_3 5$	$2 *_4 5$
$8 *_3 7$	$8 *_4 7$
$7 *_3 8$	$7 *_4 8$
$15 *_3 100$	$15 *_4 100$
$100 *_3 15$	$100 *_4 15$

In the $*_3$ operation, does the *order* of the numbers affect the number produced? It is in fact easy to see from the way in which $*_3$ was defined that the ordered pair (a,b) will always produce the same number as the ordered pair (b,a). We may state this formally as follows:

For every whole number *a* and every whole number *b*,

$$a *_3 b = b *_3 a.$$

This is a statement of the *commutative property* of $*_3$, and we say that $*_3$ is a commutative operation. (You will recall the use of the word "commutative" from Chapter 0 and 1.)

From the list above, we see at once that the $*_4$ operation is *not* commutative. This conclusion follows from the fact that $5 *_4 2 \neq 2 *_4 5$, even without looking at the rest of the examples. We say that "$5 *_4 2 \neq 2 *_4 5$" is a *counterexample*: that is, it is an example counter to (or against) the commutativity of $*_4$. It is often easy to show that something is false simply by finding one counterexample.

Again referring to the operations of Section 2.5, tell what number each of the following expressions names:

$(2 *_6 3) *_6 5$	$(2 *_4 3) *_4 5$
$2 *_6 (3 *_6 5)$	$2 *_4 (3 *_4 5)$
$4 *_6 (1 *_6 3)$	$4 *_4 (1 *_4 3)$
$(4 *_6 1) *_6 3$	$(4 *_4 1) *_4 3$
$(2 *_6 2) *_6 6$	$(2 *_4 2) *_4 6$
$2 *_6 (2 *_6 6)$	$2 *_4 2\ *_4 6)$

From these examples, we see, for instance, that $(2 *_4 3) *_4 5 = 2 *_4 (3 *_4 5)$; that is, the result is the same whether the *last two* numbers or the *first two* numbers are "associated" by parentheses. The same is true for the other examples using the $*_4$ operation. It is in fact true no matter what three numbers are selected. We may state this as follows:

For every whole number *a*, every whole number, b and every whole number *c*,

$$(a *_4 b) *_4 c = a *_4 (b *_4 c).$$

This is a statement of the associative property of $*_4$, and we say that $*_4$ is an associative operation. (Recall from Chapter 1 that addition and multiplication in the finite systems discussed there were also associative.)

Question: From the list above, can you find a *counterexample* showing that $*_6$ is not an associative operation?

Next, tell what number each of the following expressions names:

$5 *_3 0$	$5 *_6 0$
$0 *_3 5$	$0 *_6 5$
$142 *_3 0$	$142 *_6 0$
$0 *_3 142$	$0 *_6 142$
$55 *_3 0$	$55 *_6 0$
$0 *_3 55$	$0 *_6 55$

How may the "behavior" of the number 0 under the $*_3$ operation be described? Do you see (from the way the $*_3$ operation was defined, not just from the illustrations above) that *for any whole number a*, $*_3$ assigns to the pair (a,0) *the number a itself*? It also assigns a to the pair (0,a). In other words, for any whole number a, $a *_3 0 = a$, and $0 *_3 a = a$. We often put these statements together in the following way:

For every whole number *a*, $a *_3 0 = 0 *_3 a = a$.

This statement says that 0 is an *identity element* for $*_3$. (When 0 is put in a pair with any number a, $*_3$ produces "identically" the same number a.)

Question: Can you give a counterexample to show that 0 is *not* an identity element for $*_6$?

Is there a number which is an identity element for $*_6$?

Let us look again at the operational system (Z_3, +) studied in Chapter 1. The appropriate table is shown below:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The table shows clearly that + is an operation on the set Z_3. Why? Furthermore, we see that 0 is an identity element. Now, note the following assignments:

$$(0,0) \longrightarrow 0$$
$$(1,2) \longrightarrow 0$$
$$(2,1) \longrightarrow 0$$

What we have done is to list the ordered pairs of numbers which are assigned the identity element 0. And, as you recall, the numbers in such a pair are called inverses under +. The numbers 2 and 1 are inverses, since $2 + 1 = 0$ and $1 + 2 = 0$: we also say that "2 is the inverse of 1" and "1 is the inverse of 2." This is the same way we shall use the word "inverse" when speaking of an operation; two elements are inverses for an operation if together they produce the identity element of the operation.

Question: What is the inverse of 0 in (Z_3, +)?

In this section, we have looked at four important properties of operations: commutativity, associativity, identity element, and inverse elements. Let us now try to summarize them by using the "*" symbol to denote an operation on a set S.

1.) Commutativity.

For every a in S, and every b in S,

$$a * b = b * a$$

2.) Associativity.

* is associative if:

For every a in S, every b in S, and every c in S,

$$(a * b) * c = a * b * c)$$

3.) Identity.

Suppose e is an element of the set S. e is an identity element for (S, *) if:

For every a in S, $a * e = e * a = a$

4.) Inverse.

Suppose e is an identity element of *. Then a and b are inverses of each other if:

$$a * b = b * a = e.$$

In the exercises, you will have a chance to apply these definitions to many different operations. This should help you to see clearly what they mean.

2.10 Exercises.

1. Tell what whole number is named by each of the following. Warning: Some of the expressions do not name any whole number at all.

(a) 82 + 517
(b) 517 + 82
(c) 517 − 82
(d) 82 − 517
(e) 82 × 517
(f) 517 × 82
(g) 816 ÷ 8
(h) 8 ÷ 816

2. Which of the following are true for every whole number a, every whole number b?

(a) $a + b = b + a$
(b) $a - b = b - a$
(c) $a \cdot b = b \cdot a$
(d) $a \div b = b \div a$

3. Which of the following statements are true?
   (a) Addition of whole numbers is commutative.
   (b) Subtraction of whole numbers is commutative.
   (c) Multiplication of whole numbers is commutative.
   (d) Division of whole numbers is commutative.
4. (a) Are there any whole numbers a and b for which $a - b = b - a$?
   (b) Are there any whole numbers a and b for which $a \div b = b \div a$?
5. Look again at the six operations defined in Section 2.5. Which of these are commutative operations? (Give a counterexample for each operation which is not commutative.)
6. Tell what whole number is named by each of the following:

(a) (12 + 6) + 2
(b) 12 + (6 + 2)
(c) 12 − (6 − 2)
(d) (12 − 6) − 2
(e) (12 × 6) × 2
(f) 12 × (6 × 2)
(g) (12 ÷ 6) ÷ 2
(h) 12 ÷ (6 ÷ 2)

7. Which of the following are true for every whole number a, every whole number b, and every whole number c?

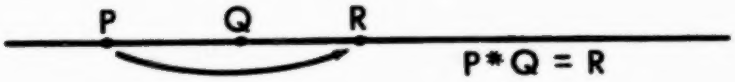
(a) $(a + b) + c = a + (b + c)$

(b) $(a - b) - c = a - (b - c)$

(c) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(d) $(a \div b) \div c = a \div (b \div c)$

8. Which of the following statements are true?
   (a) Addition of whole numbers is associative.
   (b) Subtraction of whole numbers is associative.
   (c) Multiplication of whole numbers is associative.
   (d) Division of whole numbers is associative.
9. (a) Are there any whole numbers a, b, and for which $(a - b) - c = a - (b - c)$?
   (b) Are there any whole numbers a, b, and c for which $(a \div b) \div c = a \div (b \div c)$?
10. Look again at the six operations defined in Section 2.5. Which of these do you think are associative operations? Try to find a counterexample for each operation which is not associative.
11. (a) Evaluate the following:
    $15 + 0$; $0 + 15$; $312 + 0$; $0 + 312$.
    (b) Name an identity element for addition of whole numbers. Is there more than one identity element?
    (c) Evaluate the following:
    15×1; 1×15; 312×1; 1×312.
    (d) Name an identity element for multiplication of whole numbers. Is there more than one identity element?
12. Notice that although $2 - 0 = 2$, it not true that $0 - 2$ yields 2. Therefore, 0 is not an identity element for subtraction of whole numbers. (Look again at the definition of identity element if you do not see why this is the case.) Is there an identity element for division of whole numbers?
13. Construct an operational table for $(Z_6, +)$.
    (a) Is + a commutative operation? (How does the table show this?)
    (b) Is + an associative operation? (Is there a counterexample?)
    (c) Is there an identity element for +?
    (d) List all pairs of numbers which are inverses for +.
14. Construct an operational table for $(Z_6, \cdot)$.
    (a) Is $\cdot$ commutative?
    (b) Is $\cdot$ associative?
    (c) is there an identity element for $\cdot$?
    (d) List all pairs of numbers which are inverses for $\cdot$.
15. Look again at exercise 16 of Section 2.4, where we introduced an operation which assigned to every pair (P, Q) of points a midpoint. Call this operation mid in this problem.
    (a) Is it true that mid Q = Q mid P for every point P and every point Q?
    (b) Is it true that (P mid Q) mid R = P mid (Q mid R) for every point P, every point Q, and every point R?
16. In this problem, we introduce a new operation on the set of pairs of points in a plane (think of a plane as simply a flat surface like the top of a desk):
    Let P and Q be two points as below. Draw a line through these two points. Then we shall define P*Q for this problem as follows:
    P*Q is the point R which is on the line through P and Q, on the "other side" of Q from P and at the same distance from it.



We say in this case that "R is the reflection of P on Q."
    (a) Is this operation commutative?
    (b) Is this operation associative?
    (c) Does the operation have an identity element?
    (d) What point would be assigned to a pair such as (P, P)?
17. The table below defines an operation Δ over the set {a, b, c}.

Δ	b	a	c
b	b	a	c
a	a	c	b
c	c	b	a

   (a) is Δ an associative operation?
   (b) is Δ a commutative operation?
   (c) Does Δ have an identity element?
   (d) If there is an identity element, list all pairs of inverse elements.
18. (a) Consider the system (W, +) – that is, addition of whole numbers.
    Does the number 8 have an inverse in this system?

Does the number 0 have an inverse in this system?

(b) Consider the system (W, ·) – that is, multiplication of whole numbers.

Does the number 8 have an inverse in this system?

Is there a number which does have an inverse in the system?

2.11 Cancellation Laws

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$(Z_5, +)$

•	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$(Z_5, \cdot)$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$(Z_4, +)$

•	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$(Z_4, \cdot)$

Suppose that two people are asked to choose an element from Z_5 without telling what number they have chosen. Each, however, is to write some true sentence about his "unknown" number. The first person calls his number a, and writes the following statement:

$$3 + a = 2.$$

The second person, without knowing what the first has written, calls his number b and writes the following:

$$3 + b = 2.$$

What conclusion can be drawn? It is apparent, from a glance at the $(Z_5, +)$ table, that a is 4 and that b is also 4, because 4 is the only number which, with 3, produces the number 2 in $(Z_5, +)$. In other words, a and b are the same number, and we may write

$$a = b.$$

There is an important idea suggested here. Notice that from the statements made by the two people, we knew the following:

$$3 + a = 3 + b,$$

since the expressions on the two sides of the "=" sign were both given as equal to 2. We were then able to conclude:

$$a = b.$$

Would we have been able to make the same conclusion if 3 + a and 3 + b had both been given as equal to 3, instead of 2? The answer is "yes", since in such a case both a and b would have to be 0. In fact, as you can verify yourself, as long as 3 + a = 3 + b, we may conclude that a and b are the same number. Thus, we write :

In $(Z_5, +)$, If $3 + a = 3 + b$, then $a = b$.

There is nothing special about the number 3 in this argument. If, for instance, we know $2 + a = 2 + b$ or that $0 + a = 0 + b$ or that $1 + a = 1 + b$ or that $4 + a = 4 + b$, we can still conclude that a and b are the same number. In other words, if c is any number in Z_5,

In $(Z_5, +)$, If $c + a$, $c + b$ then $a = b$.

This is known as the cancellation law of addition in Z_5.

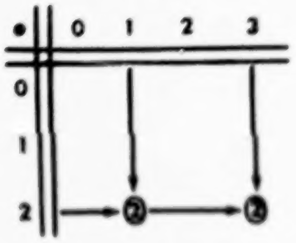
Now let us look at $(Z_4, \cdot)$. Suppose we know that a and b are two numbers in Z_4, and we know further that

$$2 \cdot a = 2 \cdot b.$$

Can we conclude that a and b are the same number? Be careful! At first, it might seem that this conclusion is justified. But look at the table for $(Z_4, \cdot)$. Do you see that

$$2 \cdot 1 = 2 \text{ and also } 2 \cdot 3 = 2?$$

This shows up clearly in the table since the number 2 appears more than once in a row:



In this case, $2 \cdot 1 = 2 \cdot 3$, but $1 \neq 3$. Hence, there is no cancellation law in $(Z_4, \cdot)$.

Next, look at the table for $(Z_5, \cdot)$. Is there any number which appears more than once in any row of the table? Surely 0 does, since every entry in the first row is "0". So, even if we know

$$0 \cdot a = 0 \cdot b,$$

we cannot conclude that $a = b$. (For example, a might be 2, and b might be 3; yet $0 \cdot 2 = 0 \cdot 3$.) However, no number except 0 appears more than once in any row. Therefore,

In $(Z_4, \cdot)$, if $c \neq 0$ and $c \cdot a = c \cdot b$, then $a = b$.

So we have a cancellation law in $(Z_5, \cdot)$ provided that we take care of the exception with zero.

Question: Examine the table for $(Z_4, +)$. Is there a cancellation law in this system? Is there an easy way to tell from the table?

In the following examples, we investigate some cancellation laws with the whole numbers. Specifically, we shall work with the systems $(W, +)$ and $(W, \cdot)$.

Example 1. If $4 + a = 4 + b$, is it true that $a = b$? The answer, of course, is "yes". Recall that in the addition table for whole numbers, no number appears more than once in any row (though the table goes on without end).

Example 2. If $a + 4 = b + 4$, does $a = b$? The answer again is "yes". In fact, since + is a commutative operation over W, this is essentially the same as Example 1.

Example 3. If $4 \cdot a = 4 \cdot b$, does $a = b$? Once again, the answer is "yes". For instance, if $4 \cdot a = 20$, and $4 \cdot b = 20$, then both a and b are 5. Because of commutativity, we can also say, "If $a \cdot 4 = b \cdot 4$, then $a = b$."

Example 4. If $0 \cdot a = 0 \cdot b$, does $a = b$? NO! Recall that in the multiplication table for whole numbers, "0" is the entry in every cell of the first row. Thus, $0 \cdot 2 = 0 \cdot 58$, since both products are 0; but $2 \neq 58$.

From examples such as these, it seems reasonable to formulate the following cancellation laws for addition and multiplication of whole numbers.

For every whole number a, every whole number b, every whole number c,

$$\text{if } c + a = c + b \text{ the } a = b$$

For every whole number a, every whole number b, and every whole number $c \neq 0$,

$$\text{if } c \cdot a = c \cdot b, \text{ then } a = b.$$

Are you clear as to why we require $c \neq 0$ in the cancellation law for multiplication of whole numbers? (If not, see Example 4 above).

Of course, since addition and multiplication of whole numbers are commutative operations, these cancellation laws could just as well have been stated in the following way:

$$\text{If } a + c = b + c, \text{ then } a = b.$$

$$\text{If } a \cdot s = b \cdot c \text{ (and } c \neq 0\text{), then } a = b$$

We have now seen several systems in which cancellation laws are possible, and at least one, $(Z_4, \cdot)$, where there is no cancellation law. The notion of a cancellation law in an operational system may be defined in general as follows:

Definition: If $(S, *)$ is an operational system, we say that there is a cancellation law in $(S, *)$ provided that the following is true for every a, b, and c in set S:

$$\text{If } a * c = b * c, \text{ then } a = b.$$

2.12 Exercises.

1. Suppose that a and b are two whole numbers such that

   $5 \cdot a = 95$; and $5 \cdot b = 95$.

   What number is a ? What number is b?

2. Suppose that x and y are two whole numbers such that

   $x + 79 = 112$; and $y + 79 = 112$.

   What number is x? What number is y?

3. Suppose a, b, and c are whole numbers. What conclusions can you draw from the following?

   (a) $c + a = c + b$

   (b) $c \cdot a = c \cdot b$, where $c \neq 0$

   (c) $0 \cdot a = 0 \cdot b$

4. Consider again the "maximizing" operation over the whole numbers.

   (a) Suppose there are two whole numbers a and b such that

   $$4 \max a = 4 \max b.$$

   Can you conclude that $a = b$?

   (b) Is there a cancellation law for $(W, \max)$?

5. Let mid be the operation which assigns to every pair of points (P, 1) the midpoint. (See problem 16 of Section 2.4).
   Is there a cancellation law for this operation? That is, if P mid Q = P mid S, where P, Q, and S are points, can you be sure that Q and S are the same point?

6. For which of the following systems is there a cancellation law?

   (a) $(8, +)$ (b) $(Z_9, +)$ (c) $(Z_8, \cdot)$ (d) $(Z_9, \cdot)$

7. From which of the following statements can you conclude that $a = b$?

   (a) $2 + a = 2 + b$

   (b) $0 + a = 0 + b$

   (c) $2 \cdot a = 2 \cdot b$

   (d) $0 \cdot a = 0 \cdot b$

   (e) $2 \max a = 2 \max b$

(f) $2^a = 2^b$ (where $\underline{a}$ and $\underline{b}$ are not zero)

(g) $a^2 = b^2$

(h) $2 + a = 2 + b$ in $(Z_3, +)$

(i) $2 \cdot a = 2 \cdot b$ in $(Z_3, \cdot)$

(j) $2 + a = 2 + b$ in $(Z_4, +)$

(k) $2 \cdot a = 2 \cdot b$ in $(Z_4, \cdot)$

8. Let * be the operation which assigns to an ordered pair of points (P, Q) in a plane the reflection of P on Q. (See exercise 16 of Section 2.10).
   Is there a cancellation law for this operation?
9. The following table defines an operation on the set $\{a, b, c\}$
   Is there a cancellation law for this operation?

*	a	b	c
a	a	b	c
b	b	c	b
c	c	a	b

10. Make up two new operations over the set W of whole numbers, so that one of the operations has a cancellation law and the other one does not.
11. The sum of two even whole numbers is an even number. We might abbreviate this statement as:

    even + even = even.

    In the same way, we can say the following

    odd + odd = even

    even + odd = odd

    odd + even = odd

    Now we consider the set S = {even, odd} having two elements, we can construct the following operational table:

+	even	odd
even	even	odd
odd	odd	even

   (a) In (S, +), is + associative?
   (b) Is + commutative?
   (c) Is there an identity element?
   (d) Does each element have an inverse?
   (e) Does the system (S, ·) have a cancellation law?
12. Using the set S = {even, odd} from problem 11, construct an operational table for the system (S, ·).
   (a) In (S, ·), is · associative?
   (b) Is · commutative?
   (c) Is there an identity element?
   (d) Does each element have an inverse?
   (e) Does the system (S, ·) have a cancellation law?

2.13 Two Operational Systems

Let S be the set

$$\{6, 8, 2, 4\},$$

a subset of the set of even whole numbers. We are going to introduce an operation on this set which we shall denote by the symbol "$\odot$", since it is closely related to ordinary multiplication of whole numbers. To begin with an illustration, consider the problem of making an assignment to the ordered pair (8, 4). The ordinary product of 8 and 4 is 32, but we shall keep only the last digit, and write

$$8 \odot 4 = 2.$$

As another example, the ordered pair (2, 8) shall be assigned the number 6 by the $\odot$ operation, since the ordinary product of 2 and 8 is 16, and the last digit of the numeral "16" is "6".

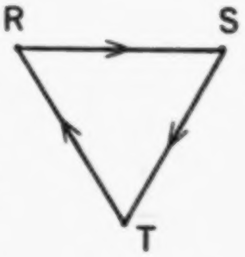
As you can see, the $\odot$ operation makes its assignments on the basis of certain digits; so we can call it digital multiplication. Printed below is an operational table showing the assignments for all ordered pairs of elements of S, under digital multiplication.

⊙	6	8	2	4
6	6	8	2	4
8	8	4	6	2
2	2	6	4	8
4	4	2	8	6

From the table, you can see that we were justified in calling $\odot$ an operation on S. In the exercises that follow, you will be asked to investigate some of the properties of the operational system (S, $\odot$).

For the second operational system, we use the set P of all points in a plane. For example, R and S are two points in the plane of this sheet R. .S of paper. If we are to have a binary operation on this set P, we must be able to assign to every ordered pair of points such as (R, S)some particular point of the plane. Let us agree to make the assignment in the following way:

Draw a segment connecting R and S. Then find a point T so that when a segment is drawn connecting R and T, and another segment is drawn connecting S and T, you have a triangle all of whose sides have the same length. Also, if you trace the trangle going from R to S to T, you are moving in the same "direction" as the hands of a clock.



$(R,S) \longrightarrow T$

What assignment could we make to an ordered pair such as (R, R) consisting of two equal points? If in such a case we agree simply to assign the point R itself, then we are able to make an assignment – and only one assignment – to every pair of points. We have an operation; and since a triangle helped us to define this operation, let us call the operation "tri". Thus, for the points above, we have

$$R \text{ tri } S = T.$$

Is S tri R the same as R tri S? In the exercises that follow, you will have a chance to answer questions such as this about the system (P, tri).

2.14 Exercises.

Questions 1 – 6 are about the operational system $(S, \odot)$ explained in the text.

1. (a) Is $\odot$ commutative? If not, give a counterexample.
   (b) How does the pattern of the operational table show that your answer in (a) is correct?
2. (a) Compute the following:

   $8 \odot (6 \odot 4)$; $(8 \odot 6) \odot 4$

   (b) Is $\odot$ associative? Is there any way you can tell without testing every possible case?
3. (a) Is there an identity element for $(S, \odot)$?
   (b) Is there more than one identity element for $(S, \odot)$?
4. (a) What is the inverse element of 8 in $(S, \odot)$?
   (b) What number is its own inverse in $(S, \odot)$?
5. (a) If $2 \odot a = 2 \odot b$, what can you conclude about a and b?
   (b) Is there a cancellation law in $(S, \odot)$? How can this question be answered by inspection of the table?
6. Solve the following open sentences in $(S, \odot)$.

   (a) $x \odot 2 = 6$
   (b) $2 \odot x = 2$
   (c) $x \odot x = 6$
   (d) $x \odot x = 8$
   (e) $(x \odot x) \odot x = 2$
   (f) $x \odot (x \odot x) = 2$

Question 7 – 11 refer to the operational system (P, tri) discussed in the text.

7. Is "tri" commutative? If not, give a counterexample.
8. Is "tri" associative? (Try at least two different cases.)
9. Is there an identity element for (P, tri)? Is there more than one identity element?
10. Does every point have an inverse in (P, tri)? Defend your answer.
11. Is there a cancellation law in (P, tri)?
12. (a) Does the system $(S, \odot)$ have any properties which (P, tri) does not have?
    (b) Does the system (P, tri) have any properties which $(S, \odot)$ does not have?

2.15 What Is a Group?

In this chapter, we have studied many different operational systems, and we have called attention to such properties as associativity, identity elements, and inverse elements. Because operational systems which possess these three properties play an important role in mathematics, we give the special name group to any such system. That is, if (S, *) is an operational system such that

1.) * is associative;
2.) There is an identity element; and
3.) Each element has an inverse

then (S, *) is said to be a group.

Questions: Is $(Z_3, +)$ a group?

Is $(W, +)$ a group?

Notice that the operation in a group does not have to be commutative. However, it may be, and if it is, the group is called a commutative group.

Questions: Is $(Z_3, +)$ a commutative group?

Is $(W, +)$ a commutative group?

2.16 Exercises.

Decide which of the following are commutative groups. Remember that there are four necessary propertiies, and each must be verified.

1. $(Z_4, +)$
2. $(Z_4, \cdot)$
3. (W, max)
4. $(S, \odot)$, where $S = \{6, 8, 2, 4\}$ and $\odot$ is digital multiplication (see Section 2.13)
5. (P, tri) (see Section 2.13).

6. Consider the set of rotations of a square (see Problem 4 of section 1.12) and the operation "followed by." Do this set and this operation form a group? If so, is it a commutative group?

2.17 Summary

1. An operation on a set S is an assignment of one and only one element of S to every ordered pair of elements of S. If an operation assigns c to the ordered pair (a, b), we may show the assignment as:

$$(a, b) \longrightarrow c.$$

If a symbol such as "*" is used to identify the operation, the assignment may be shown as:

$$a * b = c.$$

When * is an operation on set S, we denote the operational system by the pair (S, *). Frequently parentheses are used to show which two elements in an expression are to be taken as an ordered pair.

2. If a and b are elements of S, and (S, *) is an operational system, then a sentence such as

$$a * x = b$$

is an open sentence in the system. Any element of S which, when substituted for x, gives a true statement, is called a solution of the open sentence.

3. There are certain properties of operations which are important. For example, if (S, *) is an operational system and we let a, b, and c represent elements of S, then
    - * is commutative if $a * b = b * a$ for every a and b:
    - * is associative if $(a * b) * c = a * (b * c)$ for every a, b, and c:
    - e is an identity element of (S, *) if $e * a = a * e = a$ for every a:
    - a and b are inverse elements in (S, *) if $a * b = b * a = e$ where e is an identity.

If in this system $a * c = b * c$ implies $a = b$, for every a, b, and c, we say that (S, *) has a cancellation law. In systems that are not commutative, there may be "right" and "left" cancellation laws.

4. A group is any operational system (G, *) in which * is associative, there is an identity element, and every element has an inverse. If * is also commutative, then the group (G, *) is called a commutative group.

5. (W, +) and (W, ·) are two important systems involving the whole numbers. These operational systems have the following properties:

In (W, +), + is associative:
+ is commutative:
there is an identity element, 0:
there is a cancellation law.
In (W, ·) · is associative:
· is commutative:
there is an identity element, 1:
if $c \neq 0$, $a \cdot c = b \cdot c$ implies $a = b$.

2.18. Review Exercises

1. Tell what number is assigned to the ordered pair (7, 2) in each of the following systems:
    (a) (W, +) (b) (W, ·) (c) (W, max)
    (d) $(Z_{12}, +)$ (e) $(Z_{12}, \cdot)$

2. List all pairs which are assigned the number 4 in each of the following systems:
    (a) (W, +) (b) (W, ·) (c) (W, max)
    (d) $(Z_{12}, +)$ (e) $(Z_{12}, \cdot)$

3. Tell what whole number (if any) is named by each of the following.
    (a) 867 + 245
    (b) 245 + 867
    (c) 867 − 245
    (d) 245 − 867
    (e) 867 max 245
    (f) 245 max 867
    (g) 87 · 5
    (h) 5 · 87
    (i) 87 ÷ 5
    (j) 5 ÷ 87
    (k) 3^3
    (l) 3^4
    (m) 4^3

4. Which of the following are operations on the set W of whole numbers?
    (a) addition
    (b) multiplication
    (c) subtraction
    (d) division
    (e) maximizing
    (f) raising to a power

5. Which of the following statements are true for every whole number a, every whole number b, and every whole number c?
    (a) $a + b = b + a$
    (b) $a \cdot b = b \cdot a$
    (c) $a - b = b - a$
    (d) $a \div b = b \div a$
    (e) a max b = b max a
    (f) $a^b = b^a$

6. Find the number named by each of the following, if a is 12, b is 6, and c is 2.
    (a) $(a + b) + c$
    (b) $a + (b + c)$

(c) $(a - b) - c$

(d) $a - (b - c)$

(e) $(a \cdot b) \cdot c$

(f) $a \cdot (b \cdot c)$

(g) $(a \div b) \div c$

(h) $a \div (b \div c)$

(i) $(a \max b) \max c$

(j) $a \max (b \max c)$

7. Find the number named by each of the following if a is 4, b is 2, and c is 3.

(a) $(a^b)^c$ (b) $a^{(b^c)}$

8. Which of the following are associative?
   (a) addition of whole numbers
   (b) division of whole numbers
   (c) subtraction of whole numbers
   (d) multiplication of whole numbers
   (e) maximizing with whole numbers
   (f) raising to a power with natural numbers

9. "Averaging" is not an operation on the whole numbers, but assignments can be made to certain pairs. Let "aVb" mean "the average of a and b."

(a) What is 8 V (12 V 20)?

(b) What is (8 V 12) V 20?

(c) Is "averaging" associative?

10. Find what number each of the following names.

(a) $((6 + 7) \cdot 3) + 16$

(b) $((9 \cdot 5) \max 46) + 156$

(c) $100 \cdot ((2^3) + 17)$

(d) $((5 + 7) \cdot (3 + 17)) \cdot 10$

(e) $((5 \max 7) \cdot 8) + ((5^3) + 3)$

11. Find all whole number solutions of the following open sentences. If there is no whole number solution, say so.

(a) $156 + x = 217$ (k) $a^3 = 8$

(b) $89 + a = 89$ (l) $3^a = 8$

(c) $89 + a = 88$ (m) $1^a = 1$

(d) $a \cdot 14 = 98$ (n) $1^a = 2$

(e) $a \cdot 14 = 99$ (o) $a + a = 100$

(f) $14 \cdot a = 14$ (p) $a \cdot a = 100$

(g) $14 \cdot a = 0$ (q) $a^a = 100$

(h) $4 \max n = 4$ (r) $n^n = 27$

(i) $4 \max n = 5$ (s) $a \max a = 100$

(j) $4 \max n = 3$ (t) $(2 \max a)^2 = 4$

12. Find what number is named by each of the following if $a = 2$ and $b = 5$.

(a) $a^3 + 2$ (g) $a + b^2$

(b) $2a^3$ (h) $2a^3 + 5$

(c) $(2a)^3$ (i) $2 \cdot (a^3 + 5)$

(d) $(a + b)^2$ (j) $(a \max b)^2$

(e) $a^2 + b^2$ (k) $a \max (b^2)$

(f) $a^2 + (2 \cdot a \cdot b) + b^2$

13. If each of the following is taken to be a true statement about the whole numbers a and b, from which can we conclude that $a = b$?

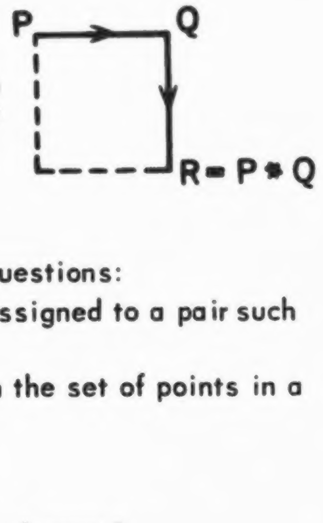
(a) $5 + a = 5 + b$ (e) $3 \max a = 3 \max b$

(b) $0 + a = 0 + b$ (f) $a^3 = b^3$

(c) $5 \cdot a = 5 \cdot b$

(d) $0 \cdot a = 0 \cdot b$

14. Consider all ordered pairs of points in a plane. If (P, Q) is an ordered pair of points, let P*Q be found in the following way: Take P and Q as corners of a square, and let R be the third corner of the square if you move in a "clockwise" way from P to Q to R. (See picture at right.) Then P * Q = R.



Answer the following questions:
   (a) What point can be assigned to a pair such as (Q, Q)?
   (b) Is * an operation on the set of points in a plane?
   (c) Is * commutative?
   (d) Is * associative?
   (e) Is there an identity element?
   (f) Is there a cancellation law?

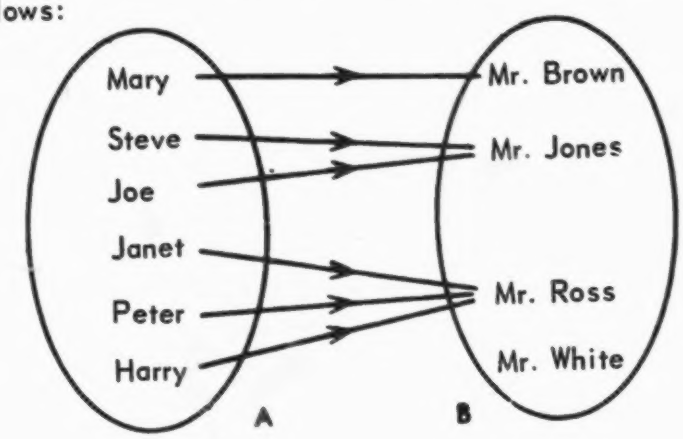
15. Take the set $S = \{0, 1\}$, and construct an operational table for every possible binary operation on S.

CHAPTER 3: Mathematical Mappings

3.1 What is a Mathematical Mapping?

To most people the word "map" suggests the kind of picture used to represent a geographical region. In such a map each point represents, or is the image of, a precise geographical location. For instance, a road map of New Jersey assigns to each major location in New Jersey exactly one point on the map. Mathematicians have used such a scheme of associating one thing with another to fashion a very useful mathematical tool which, naturally enough, is called a mathematical mapping. We begin our study of mathematical mappings with an example.

Consider a set A of children, {Mary, Steve, Joe, Janet, Peter, Harry}, and a set B of men who are fathers; {Mr. Brown, Mr. Jones, Mr. Ross, Mr. White}. We can show the actual relation by a diagram as follows:



As you see, the lines connecting child and father are given direction. They indicate, for instance, that Mary's father is Mr. Brown. You see that each child has only one father. This is what we mean when we say that the relation "father of" is a mapping which maps the set A into, or to, the set B. The person at the tip of each arrow is called the image of the first. Thus, Mr. Jones is the image (mathematically speaking) of Joe, and in the mapping each element of set A has exactly one image.

Our example had three ingredients, and you should look for them in each mapping.

1. Set A. The set of objects that may be assigned images.
2. Set B. The set from which images are chosen.
3. A rule or clear method by which one assigns to each object in A exactly one image.

Let us look at a second example. We start with A = {2, 3, 12, 7, 4}, a set of numbers. Let the second set be the set of natural numbers, and let the rule that assigns images be: to find an image multiply each number in A by 3. This mapping can be displayed as follows:

$$\begin{tikzcd} 2 \arrow[d] & 3 \arrow[d] & 12 \arrow[d] & 7 \arrow[d] & 4 \arrow[d] \\ 6 & 9 & 36 & 21 & 12 \end{tikzcd}$$

We call the set of first objects in a mapping the domain of the mapping. The set of images we call the range of the mapping. What is the domain of this mapping? What is its range? What is the domain of the mapping in our first example? What is its range?

Let us consider a third example of a mapping with domain N = {1, 2, 3, 4, . . .} and the rule: the image is 3 times a number in N. We recognize that the range is, {3, 6, 9, 12, . . .}, that is, the set consisting of the multiples of 3.

It is convenient to write the rule of this mapping: n → 3n, in which n represents a natural number. What does 3n represent? We can read "n → 3n" in any one of the following ways.

(1) The image of n is 3n
(2) n is mapped onto 3n
(3) to n is assigned 3n

What is the rule in our first example of a mapping? The rule in the second example?

Now let us look at a fourth example. We start with W = {0, 1, 2, 3, . . .}. To assign images to elements of W we use the rule n → 1/2n – 3. By this rule, 8 → (1/2 · 8) – 3 = 4 – 3 = 1, 12 → (1/2 · 12) – 3 = 6 – 3 = 3 and 24 → (1/2 · 24) – 3 = 12 – 3 = 9. But we quickly find that some whole numbers are not assigned images by this rule. For instance, 9 → (1/2 · 9) – 3 = 4-1/2 – 3 = 1-1/2, which is not a whole number. Also, 2 → (1/2 · 2) – 3 = 1 – 3, for which we can give no answer at present.

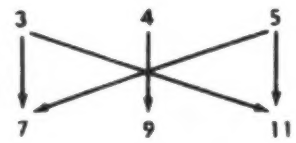
Do you see that our rule assigns a whole number as an image to just those whole numbers in the set of even whole numbers starting with 6? Hence, n → 1/2n – 3 is the rule for a mapping of the subset of whole numbers {6, 8, 10, . . . } to W. What is the range of this mapping?

From this example we can see that giving a rule of assignment is not enough to completely define a mapping. We must be sure to check that all three ingredients of a mapping are present.

3.2 Exercises

1. Take W = {0, 1, 2, 3, . . .} for the first set in a mapping and N = {1, 2, 3, . . . } for the second set in the mapping. For the rule of the mapping take n → n + 2. What is the image of 0? 1? 20? Express the mapping by two rows of numbers and arrows to indicate image relationships.
2. Try to repeat Exercise 1, using sets W and N and the rule n → n – 2. Do 0 and 1 have images? Modify W to make this a mapping.

3. Display the mapping of the set of whole numbers into the set of natural numbers by the rule $n \rightarrow 2n + 1$. Make this display by means of the double row and arrow diagram such as we used above.
4. Explain why this display does not represent a mapping.



5. For each of the following rules display a mapping of W to W, if possible. If the rule does not define a mapping of all of W to W, modify W so that the rule does define a mapping and display the mapping.

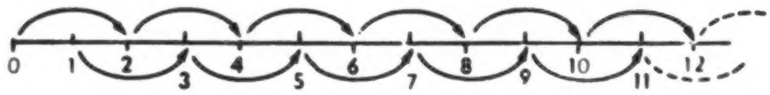
(a) $n \longrightarrow 2n$ (d) $n \longrightarrow n - 2$

(b) $n \longrightarrow 1/2n$ (e) $n \longrightarrow 2n + 3$

(c) $n \longrightarrow n + 2$ (f) $n \longrightarrow 3n - 2$

6. In this exercise you are asked to map A = {3, 4, 5} into the set of whole numbers for each of the rules given below. Tell whether the statement accompanying each rule is true or false.
   - (a) $n \rightarrow 2n$. The image of 4 is between the image of 3 and the image of 5.
   - (b) $n \rightarrow 3n + 1$. The image of 4 is the average of the images of 3 and 5.
   - (c) $n \rightarrow 3n - 1$. The images of 3, 4, 5 are consecutive numbers.
   - (d) $n \rightarrow n \times n$. The image of 4 is the average of the images of 3 and 5.
   - (e) $n \rightarrow 12 - n$. The image of 3, 4, 5 are in increasing order.
7. We can make mathematical maps for objects that are not numbers. For instance let the first set be a = {a, e, i, o, u} and let the second set be the English alphabet. Let the rule be: for the image of a letter in A take the next letter of the alphabet in alphabetical order. Thus, the image of o is p. Make a display of the map of A by this rule.
8. Show how an alphabetic code for deciphering a secret message can be put in the form of a map.
9. In this exercise, the set A contains the weights in ounces, of five letters to be mailed. Let B be the set of possible costs, in cents, of mailing letters (first class mail). Recall that post offices charge 5 cents per ounce or fractional parts of an ounce and display a map of A = {3, 4-1/2, 6.2, 1/2, 1/4} to B.

3.3 Arrow Diagrams and Mapping



Even though the last number shown on this number line is 12 you are expected to assume that we are talking here about the set of all whole numbers, and that the arrows continue in the same pattern. Look at the arrow starting at the point labelled 0, where does it end? This arrow shows that the image of 0 is 2, that is, $0 \rightarrow 2$. For the numbers shown in the diagram, the rule $n \rightarrow n + 2$ is satisfactory. Let us use this rule for all of W. Does each whole number, shown or not in the diagram, have an image? Exactly one image? Then the diagram represents a mapping. If we interpret each arrow as connecting two whole numbers we have an example of an arrow diagram of a mapping. Using the diagram, find the image of 3; of 4; of 7. Is there a number whose image is 3? 4? 7?

Does every number occur at the beginning of some arrow? If so, then every number has an image. Does every number occur at the tip of some arrow? If not, then some numbers are not images. Which numbers are not images, if any? What is the domain of this mapping? What is its range?

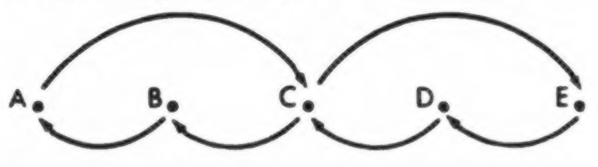
For convenience let us refer to the mapping represented in the above diagram by f. Consider this question: Does each number in the range of f serve as the image of exactly one number in the domain? Or to put it in terms of arrows, does each number that occurs at a tip of an arrow do this for exactly one arrow? If you can say yes, then f is called a one-to-one mapping. In each cases, you can easily see that there are just as many numbers in the domain as there are in the range.

On your paper copy the above number line with the same arrows, but reverse the direction of each arrow; that is, put the tip at the other end. For your diagram answer the following questions.

(1) Does every arrow end at exactly one number? If so, what bearing does this have on the question as to whether your diagram represents a mapping?

(2) Does every number occur at the beginning of some arrow? What bearing does your answer have in determining the domain of the mapping?

(3) Does every number occur at the tip of some arrow? What bearing does your answer have in determining the range of the mapping?

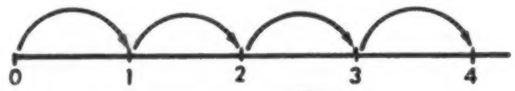


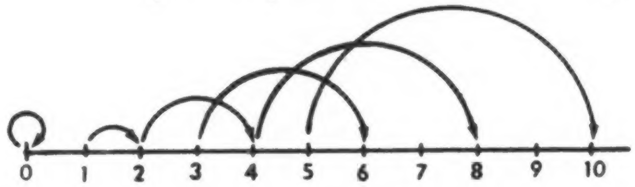
Now look at the arrow diagram above. What is the image of A? of B? Do you see any points that have more than one image? Does this diagram represent a mapping? Do you agree that if we remove one of the arrows that starts at C, then we should have a diagram for a mapping? Would this mapping be one-to-one? Why?

3.4 Exercises

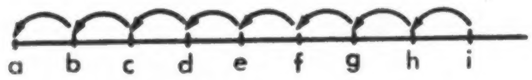
Study the arrow diagrams below and for each of them answer the following questions as far as they apply.

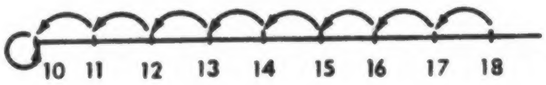
(a) Does the diagram represent a mapping? If not, why not?
(b) If it represents a mapping what is its domain? Its range?
(c) If it is a mapping and it has a rule that is easily expressed in the form $n \rightarrow ?$ state the rule.
(d) If it is a mapping, is it one-to-one?



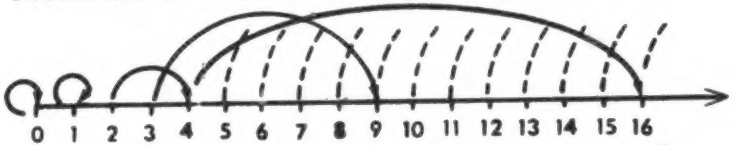


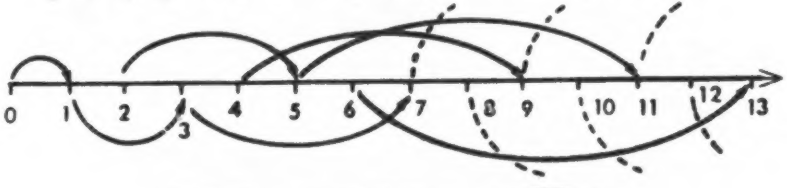
The arrow at 0 starts and ends at 0.

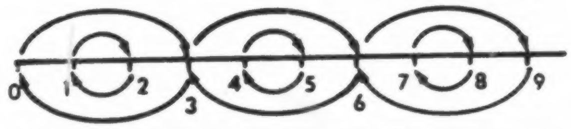




Let the dotted partial arrow indicate an infinite domain, and assume a rule that holds for the numbers shown holds for all of W.







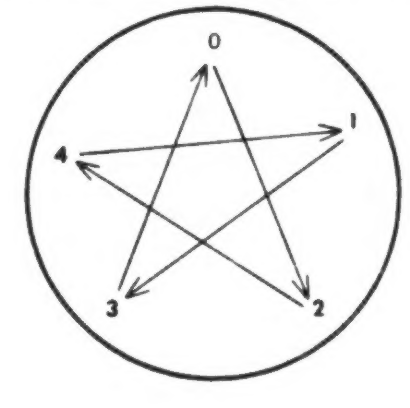
8. Make an arrow diagram for the mapping of N, or of a subset of N, to W for each of the following rules of assignment.

(a) $n \rightarrow n + 3$
(d) $n \rightarrow 2n + 1$
(b) $n \rightarrow 3 - n$
(e) $n \rightarrow n^2$
(c) $n \rightarrow 2n - 1$
(f) $n \rightarrow 1/2n$

9. For each mapping in Exercise 8 determine whether or not it is one-to-one.

3.5 Mapping of Dial Numbers

In Chapter 1 we studied finite systems consisting of dial numbers and operations on those numbers. In this section we examine mappings for such systems.



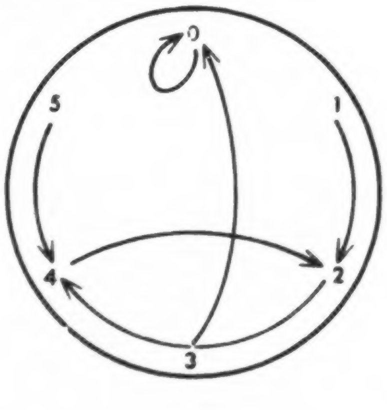
Let the domain of a mapping be the set Z_5 of numbers on a dial as shown above and let us map this set to itself by the rule $n \rightarrow n + 2$ where + means addition in (Z_5, +). What is the image of 0? of 2? of 3? What is the domain of this mapping? What is its range? For convenience let us name this mapping h.

Recall the mapping of W, into W, by the rule $n \rightarrow n + 2$. (Of course, + in this rule is ordinary addition, and recall that we named this mapping f).

Compare the answers to the following questions as each is applied first to f and then to h.

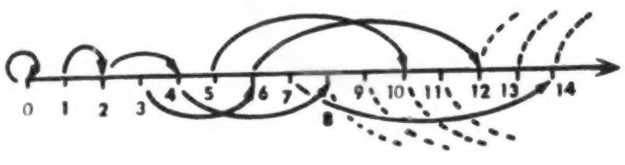
(1) Is the domain of the mapping finite or infinite? Is the range finite or infinite?
(2) Is the range of the mapping the same as the domain?
(3) Is the mapping one-to-one?

Make an arrow diagram for the mapping, (call it j), of the dial numbers shown above by the rule $n \rightarrow n - 3$. Compare it with the mapping h. Does a modification of the rule of a mapping necessarily change the mapping? Can we say h and j are the same? What do you think is meant by saying that two mappings are the same? When they are different?



Study the mapping (call it $\underline{s}$) of the set of dial numbers {0, 1, 2, 3, 4, 5} by the rule $n \rightarrow 2n$, where 2n means n + n. Explain why there are two arrows connecting 2 and 4. Notice that there are no arrows with tips at 1, 3, 5. Why do you think this is so?

The mapping $\underline{t}$, indicated below, maps W to W, by the rule $n \rightarrow 2n$. Explain why there are no arrow tips at 1, 3, 5, 7, and the other odd numbers. Answer the following questions as they apply to $\underline{s}$ and $\underline{t}$.



(a) What is the domain of the mapping? What is its range?

(b) Is the range the same as the domain?

(c) Is the mapping one-to-one?

You can answer the last question easily by noting whether or not there are numbers that serve as images for more than one number. How do arrows help you to see this?

3.6 Exercises

Study the arrow diagrams below and answer the following questions as they apply to each diagram.

(a) Does the diagram represent a mapping? If not, why not?

(b) If it represents a mapping what are the domain and range?

(c) If it is a mapping and it has an easily expressed rule in the form $n \rightarrow ?$, state the rule

(d) If it is a mapping, is it one-to-one? If not, why not?

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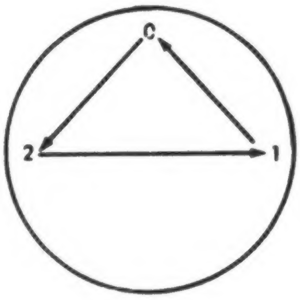

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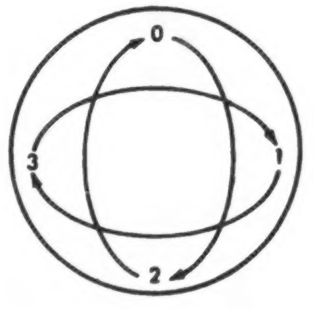

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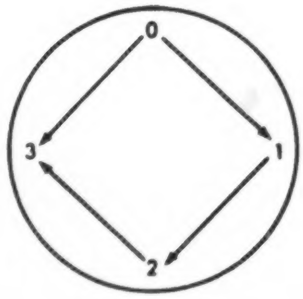

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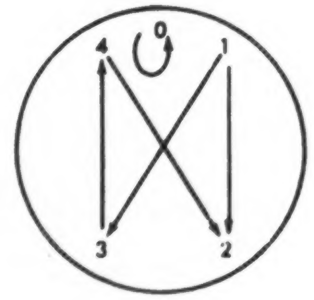

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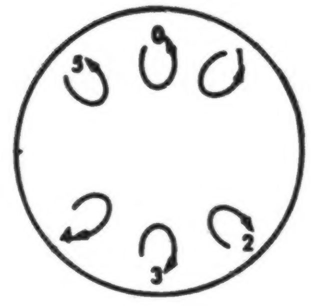

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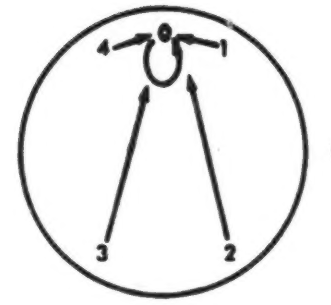

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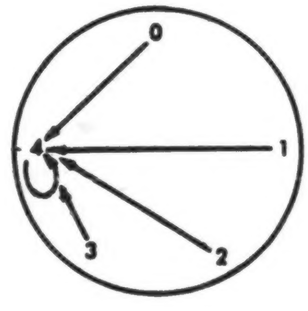

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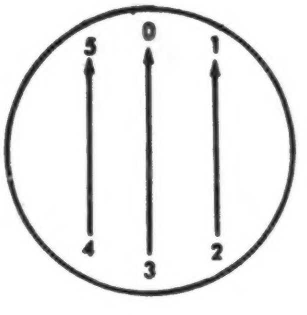


3.7 Sequences

As we have seen, the multiples of 3, that is, 3, 6, 9, 12, . . . , considered in the order written are the images in a mapping of N to N given by the rule $n \rightarrow 3n$. This is but one example of a situation we meet many times in mathematics. That is, we have a set of numbers given in an order, or an ordered set. Another example is 2, 5, 8, 11, ... In this case, as well as in the first, it is possible to think of these numbers as the range of a mapping of N to N. What is the rule for this mapping? Do you see that it is $n \rightarrow 3n-1$? As a third example consider 3/2, 5/2, 7/2, 9/2, 11/2, ... Since these are not natural numbers they are not images in a mapping of N to N. However, they are images in a mapping of N to a different set of numbers. The rule of this mapping is $n \rightarrow n + 1/2$.

These special mappings, that is, mappings whose domain is N but whose range may be in some other set, are called sequences. Below are some other sequences together with the rule of the mapping of N that determines them.

	Rule	Sequence
(1)	$n \rightarrow 2n + 1$	3, 5, 7, 9, 11, 13, 15, 17, . . .
(2)	$n \rightarrow 3n + 2$	5, 8, 11, 14, . . .
(3)	$n \rightarrow 1/2n + 2$	2-1/2, 3, 3-1/2, 4, 4-1/2, . . .
(4)	$n \rightarrow n^2$	1, 4, 9, 16, 25, . . .
(5)	$n \rightarrow n^2 - n$	0, 2, 6, 12, 20, 30, . . .

(6) $n \to 0$ is n is even 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, . . .

$n \to 1$ if n is odd

We run into some trouble if the rule is $n \to 2n - 5$ when we try to find the image of 1, for $(2 \cdot 1) - 5$ does not represent a whole number. This difficulty will be removed when we have another set of numbers (the set of integers) available.

All of these examples of sequences are infinite sets. However, by restricting the domain to a finite set of consecutive natural numbers starting with 1 we can form what we will call finite sequences. For example 3, 5, 7, 9, 11, 13, 15, 17 is a finite sequence. Its rule is $n \to 2n + 1$. What is its domain? As an interesting digression, let us try to find the sum of all its numbers. We take advantage of a trick to find this sum as follows. Write the numbers in a row, as shown below. Then write the same numbers in a second row in reverse order. It then looks like this:

3 5 7 9 11 13 15 17

17 15 13 11 9 7 5 3

Now find the sum 3 + 17, 5 + 15, 7 + 13, . . . Is each sum the same as 3 + 17, that is the sum of the first and last numbers in the sequence? How many such sums are there? Then the sum of all the numbers in both rows is 0 × 20 and the sum of the numbers in our sequence is 1/2 × 7 × 20 or 70. (Why the 1/2?) How can you reassure yourself that this result is correct?

It is natural to wonder whether this trick works for all finite sequences. You will be asked to do some experiments in the exercises that follow to decide whether or not this trick does work for all finite sequences.

3.8 Exercises

In these exercises let p represent the number of numbers in a finite sequence and let q represent the sum of its first and last numbers. Determine whether or not the sum of the p numbers in each sequence below is equal to $1/2p \times q$ for the value of p given.

1. Take the sequence with rule $n \to 2n + 1$ and $p = 6$. What is the image of 1? of 6? What are the first and last numbers in the sequence? What is q? What is $1/2pq$? Does this check with the result obtained by ordinary addition?
2. Take the sequence with rule $n \to 3n + 2$ and $p = 5$. What is the image of 1? of 5? What is q? $1/2pq$? Does this agree with the result of ordinary addition?
3. Take the sequence with rule $n \to 1/2\, n + 2$ and $p = 10$ and carry out the experiment of seeing whether or not the trick described above works. Does it work?
4. Take the sequence with rule $n \to n^2$ and $p = 6$. Does the trick work?
5. Try the experiment with the sequence with rule $n \to n^2 - n$ and $p = 4$. Does it work?
6. Try the experiment with the sequence with rule $n \to n$ and $p = 100$. Does it work.
7. Try the experiment with the sequence with rule $n \to n + 12$ and $p = 20$.
8. Without actually carrying out the experiment make a guess as to whether or not the trick works for the sequences whose rules are listed below, taking $p = 10$.

   a. $n \to 4n - 1$ b. $n \to 5n$ c. $n \to 3n$

   d. $n \to 1/2n^2$ e. $n \to 1/3n^2 + n$

   f. $n \to 4 - n$
9. Suggest a theory, without trying to prove it, which tells when this experiment works and when it does not.
10. Form a sequence of six numbers for each of the following rules.

    (a) $n \to n$ (d) $n \to 2/3n + 1$

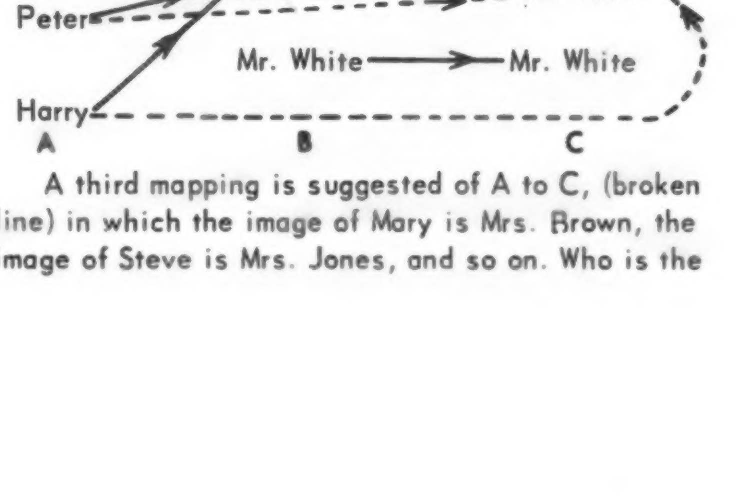
    (b) $n \to 12 - n$ (e) $n \to n^2 + 2$

    (c) $n \to 3n + 1$ (f) $n \to 1/2n^2 + 2$

3.9 Composition of Mappings

Let us recall sets A and B at the beginning of this chapter. A = {Mary, Steve, Joe, Janet, Peter, Harry} B = {Mr. Brown, Mr. Jones, Mr. Ross, Mr. White}. and consider with them the set C of wives of the men in B, where C = {Mrs. Brown, Mrs. Jones, Mrs. Ross, Mrs. White}.

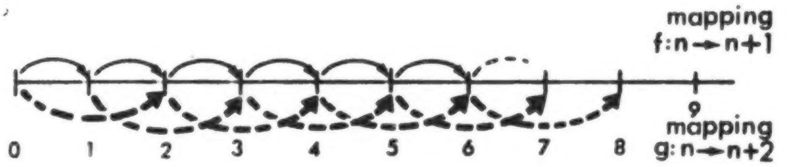
Our diagram now becomes the following, and it shows a mapping h_1, of A to B, and h_2, of B to C.



A third mapping is suggested of A to C, (broken line) in which the image of Mary is Mrs. Brown, the image of Steve is Mrs. Jones, and so on. Who is the

image of Janet? of Harry? Does every child in A have exactly one image in C?
Then indeed, this is a mapping that maps a child onto its mother. It is an example of a composition of mappings. If h_1 names the mapping from A to B, and h_2 the mapping from B to C then the mapping from A to C is called the composite of h_2 with h_1. Note the order, composite h_2 with h_1, is followed by h_2.

Let us consider a second example of a composition of mappings.



There are two mappings represented in this diagram, one named f maps W to W by the rule $n \rightarrow n+1$. The other named g maps W into W by the rule $n \rightarrow n+2$. We have indicated f with black arrows and g with broken arrows to help you distinguish one from the other. We can think of each mapping as having the effect of moving a point from one position to another. Let us follow the point at 0 and see how it is moved first by f and then from its new position by g. Mapping f moves the point from 0 to 1. We can write this, $f: 0 \rightarrow 1$ and we read it "f maps 0 onto 1", (but you can think of it as a motion). Now the point is at 1 and g moves it from 1 to 3. So we can write $g:1 \rightarrow 3$. We can write both mappings of 0 as follows: $0 \rightarrow 1 \rightarrow 3$; the net effect is $0 \rightarrow 3$. Let us see what happens to a point at 2 when we apply the same combination of mappings. First $f:2 \rightarrow 3$ and then $g:3 \rightarrow 5$. The net effect is $2 \rightarrow 5$. When we consider the combined mappings of any point of the line we see that it is moved into a position given by the rule $n \rightarrow n+3$. Do you see that there is only one mapping that maps the whole numbers into the whole numbers by this rule? Then the net effect of the two mappings f followed by g, is a mapping, call it h, of W to W with rule $n \rightarrow n+3$. So h is the composite of g with f. What would the composition of f with g mean? Do you think that the composition of g with f produces the same mapping as the composition of f with g? Do you think reversing the order of the two mappings in a composition always produces the same mapping? Let us see by forming the composition of another two mappings. Let the first of these mappings be as it was, f, the mapping of W to W by the rule $n \rightarrow n+1$, and let the second be k, the mapping of W to W by the rule $n \rightarrow 2n$.

Remember:
$f: n \longrightarrow n+1$
$h: n \longrightarrow 2n.$

First we find the image of 0 under the influence of f, and see how g effects this image. Then we reverse the order, still starting with 0.

$f:0 \rightarrow 1$ and $k:1 \rightarrow 2$. So k with $f:0 \rightarrow 2$

We reverse the order now:

$k:0 \rightarrow 0$ and $f:0 \rightarrow 1$. So f with $k:0 \rightarrow 1$

How can we find the rule of k with f? To understand this we should understand what each rule means.

$n \rightarrow n+1$ means: to get the image of a number add 1 to it.

$n \rightarrow 2n$ means: to get the image of a number double it.

Now to find the rule of k with f:

$f:n \rightarrow n+1$ (we added 1) and $k:n+1 \rightarrow 2n+2$ (we doubled)

So the rule of k with f is $n \rightarrow 2n+2$

To find the rule of f with k:
$k:n \rightarrow 2n$ (we doubled) and $f:2n \rightarrow 2n+1$ (we added 1)

So the rule of f with k is $n \rightarrow 2n+1$

Do $2n+1$ and $2n+2$ represent the same number for any value of n? Then k with f and f with k are different.

Suppose we had two mappings of N to N, one named r the other s. The rule of r is $n \rightarrow 3n+1$ and the rule of s is $n \rightarrow 2n$. To find the rule for the composite of s with r we could proceed as follows:

$r:n \rightarrow 3n+1$, $s:3n+1 \rightarrow 2\ (3n+1)$

Therefore s with $r:n \rightarrow 6n+2$

To find the rule of the composite r with s, we proceed as follows:

$s:n \rightarrow 2n$, $r:2n \rightarrow 3\ (2n)+1$

Therefore r with $s:n \rightarrow 6n+1$

3.10 Exercise

In exercises 1-8 let f, g, h be mappings from N to N with the following rules:

$f:n \rightarrow 3n$ $\quad g:n \rightarrow n+2$ $\quad h:n \rightarrow 2n+1$

1. (a) Find x in the sentence $f:1 \rightarrow x$.
   (b) Using the value of x found in a, find y such that $g:x \rightarrow y$.
   (c) Using the value of y found in b, find z such that $h:y \rightarrow z$.

2. (a) Find x such that $g:1 \to x$.
   (b) Using the value of x in a find y such that $h:x \to y$.
   (c) Using the value of y in b find z such that $f:y \to z$.
   (d) Is the value of z found in exercise 1 the same as the value found in this exercise? If not, explain why.
3. (a) Find the image of 2 under the influence of the composite of g with f.
   (b) Find the image of 2 under the influence of the composite of f with g.
   (c) Find a number whose image under g is 5.
   (d) Find a number whose image under h is 101.
4. Find a number whose image under h is 27.
   Find a number whose image under h is 32.
5. Find a number whose image under the composite of f with h is 33. It may simplify your problem to first find the rule for this composite.
6. Find a number whose image under g with f is 14.
7. Find a number such that its image under h is three times itself.
8. Find a number such that its image under the composite of g with f is four times itself.
9. Let r be the mapping of the dial numbers {0, 1, 2, 3, 4} with the rule $n \to n + 1$ and s the mapping of these dial numbers with the rule $n \to 2n$.
   a. Make an arrow diagram showing r in one color, s in a second color, r with s in a third color.
   b. Using a different diagram show s with r.
   c. Compare the drawing of r with s and s with r.

3.11 Inverse and Identity Mappings

You learned in arithmetic if you add 5 to a certain number and then subtract 5 from the sum, you end at the same number with which you began. In other words the effect of adding 5 is nullified by subtracting 5. Similarly the effect of multiplying by a number is nullified by division by that same number. In this sense addition and subtraction are a pair of inverse operations. Multiplication and division also illustrate a pair of inverse operation. This suggests the question: Is there for each mapping another, such that when one is followed by the other the effect of the first is nullified by the second? It is easy to see that this often is the case by looking at an example.

$$\begin{array}{lcccccc} W: & 0 & 1 & 2 & 3 & \ldots & n \;\ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ E: & 0 & 2 & 4 & 6 & \ldots & 2n \;\ldots \end{array}$$

Here the first set, or domain, is W and the second set is the set E = {2, 4, 6, 8, . . . }. It is easy to see, calling this mapping f, that the range of f is all of the set E. Thus, f is said to map W onto E. Now a mapping g of E to W that nullifies the effect of f must carry each image back to its source. That is $g : 0 \to 0$, $g : 2 \to 1$, and so on. In effect, then the mapping g is defined by reversing the direction of arrows in the display of f. The display for g would be

$$\begin{array}{lcccccc} E: & 0 & 2 & 4 & 6 & \ldots & n \;\ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ W: & 0 & 1 & 2 & 3 & \ldots & ? \;\ldots \end{array}$$

We are assured that this indeed defines a mapping because each element of E is assigned exactly one element in W. What is its rule? Do you see that it is $n \to n/2$?

Does g nullify the effect of f for all numbers in W? Let us find out by noting its effect on any whole number, call it n.

$$f : n \to 2n \text{ and } g : 2n \to 2n/2$$

But $2n/2 = n$

So g with f : $n \to n$

Thus we have a map f of W to E and a map g of E to W such that g with f is a map of W to W that maps each element onto itself. What can we say about f with g? Let n be any number in E, that is, any even number.

$g : n \to n/2$ (since n is even, $n/2$ is a whole number)

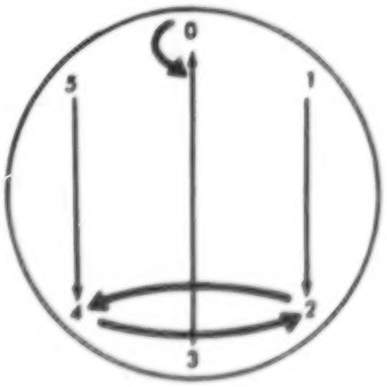
$f : n/2 \to 2 \cdot n/2$. But $2 \cdot n/2 = n$.

So f with g: $n \to n$

Thus also f with g is a map of E to E that maps each element onto itself.

Our map $f : W \to E$ that we began with is both a one-to-one and an onto mapping. We see that with these conditions there is a map g of E to W that nullifies the effect of the map f of W to E. (and also f nullifies the effect of g). Under these circumstances g is called the inverse of f (and f is the inverse of g).

Does every mapping have an inverse? Let us look at a mapping h of Z_6 to Z_6 given by the rule $n \to 2n$. The diagram below represents this mapping.



We could begin by trying to reverse the direction of each arrow in the diagram of the mapping. We see, at 4, that two arrows point to it. So reversing the direction would result in two arrows starting at 4. Then 4 would have two images! Are we going to get a mapping by reversing arrow direction? Remember that a mapping must assign exactly one image to each element in its domain.

Have you noted that in the example with an inverse, is a mapping with rule $n \rightarrow n$? Should you expect this result? Why? Such a mapping is important enough to merit its own name. We call it an identity mapping. In fact, to determine whether one mapping is the inverse of another we see if their composite (in either order) is an identity mapping.

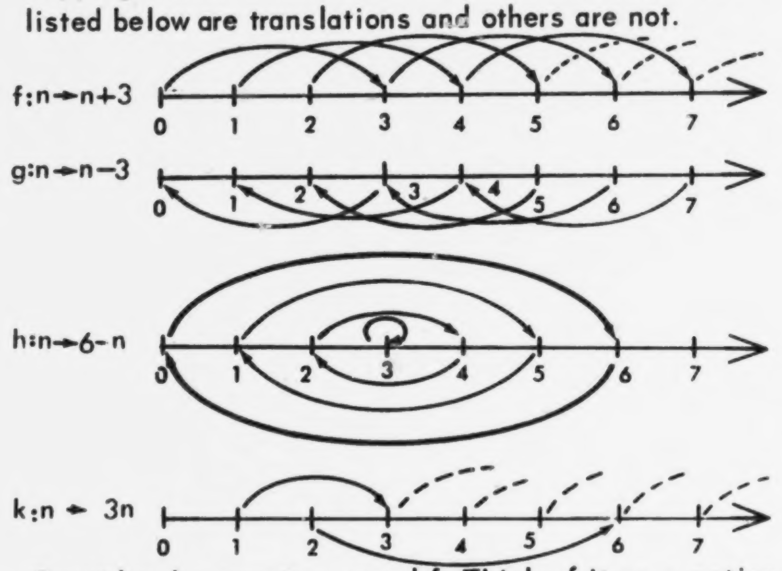
3.12 Exercises

1. Let f be the mapping of W to W with rule n 2n. Let i be the identity mapping of W to W.
   (a) Show that the composites f with i, and i with f are the same as f.
   (b) Suppose you do not know the rule for f. Do you think that the composites f with i, and i with f are the same as f? Why?
2. Make an arrow diagram of the identity mapping of W to W.
3. Make an arrow diagram of the identity mapping of the set of dial numbers $\{0, 1, 2, 3\}$ onto itself.
4. Make an arrow diagram of the mapping of N onto $\{2, 3, 4, \dots\}$ with the rule $n \rightarrow n + 2$ and on the same diagram show the inverse mapping.
5. The rule of a mapping of W to W is $n \rightarrow 3n + 2$. To find the image of a given number you perform two operations.
   (1) multiply the given number by 3. $(3n)$
   (2) add 2 to the product. $(3n + 2)$
   What is R, the range of the mapping?
   To find the number given its image, you reverse these operations and the order.
   (1) subtract 2 from the given image. $(n - 2)$
   (2) divide the difference by 3. $(n - 2)/3$
   So the rule of the inverse of the mapping of W onto R is $(n - 2)/3$.

   For each of the following rules find a domain D and a range R consisting of whole numbers, which constitute a mapping of D onto R. Then find the rule of the inverse mapping of R onto D.
   (a) $n \rightarrow 2n + 1$ (b) $n \rightarrow 3n - 2$
   (c) $n \rightarrow n - 2$ (d) $n \rightarrow n + 2$
   (e) $n \rightarrow 48n + 25$ (f) $n \rightarrow 8n + 1800$
6. Make an arrow diagram of the mapping of the set of dial numbers $\{0, 1, 2, 3, 4\}$ to itself, for each of the following rules and determine whether it has an inverse. If so find its rule.
   a. $n \rightarrow n + 1$ b. $n \rightarrow n - 3$
   c. $n \rightarrow 3$ d. $n \rightarrow 2n + 1$
7. A remark was made in parenthesis that the composite of a mapping and its inverse in either order is the identity.
   a. Show by an illustration that this is indeed true. If you wish you might use a mapping with the rule $n \rightarrow 2n + 5$.
   b. Devise a convincing argument that the composition of a one-to-one onto mapping and its inverse is commutative.

3.13 Translations Along a Line

Among the various kinds of mappings between subsets of W which we have looked at so far there are some that deserve special attention. One of these is called a translation. According to the dictionary one meaning attached to this word is a motion in which every point is moved the same distance in the same direction. This is a good description of the special mapping called a translation. Some of the mappings listed below are translations and others are not.



Consider the mapping named f. Think of it as a motion of an object starting at a point in the line and ending at the image point. f then moves the object from 0 to 3, or from 2 to 5, or from 105 to 108. The motion is exactly the same regardless of the starting point. It is always a motion of 3 units to the right. f is thus an example of a translation. Its domain is W and it is a mapping of W onto the set of whole numbers $\{3, 4, 5, 6, \dots\}$ with the rule $n \rightarrow n + 3$.

We can also think of g as moving 3 units to the left. Can the points 0, 1, and 2 be starting points for such a motion? Do you see that every whole number 3 or greater can represent a starting point? g is again a translation. Its domain is the subset of the whole numbers $\{3, 4, 5, 6, \dots\}$ and it is a mapping of $\{3, 4, 5, 6, \dots\}$ onto W with rule $n \rightarrow n - 3$. We note that h and k are not translations. You should check to see why this is the case.

We conclude that a translation must have a rule of the form $n \rightarrow n + r$ or $n \rightarrow n - r$, where r is a whole number. It is also easy to see that a translation is a one-to-one mapping. Also, we note that a translation maps its first set onto its second wet. The following examples may help to make the idea of a translation clear to you.

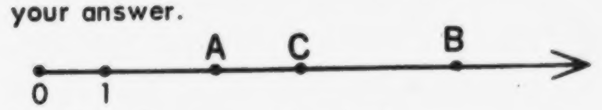
(1) The translation with rule $n \rightarrow n + 5$ maps W onto the set of whole numbers $\{5, 6, 7, \dots\}$
(2) The translation with rule $n \rightarrow n - 4$ maps the set of whole numbers $\{4, 5, 6, 7, \dots\}$. onto W.
(3) The translation with rule $n \rightarrow n + 1$ maps the set of whole numbers $\{6, 7, 8, \dots\}$ onto the set of whole numbers $\{7, 8, 9, \dots\}$.

Some questions come to mind about translations. We list them here and you will be asked to answer them in the exercises.

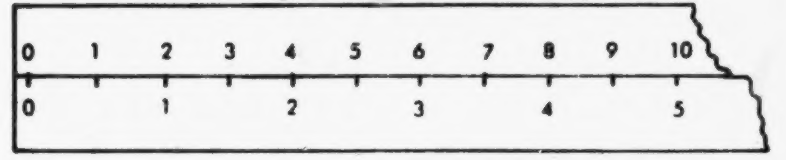
(1) Does every translation have an inverse?
(2) Is the inverse of a translation also a translation?
(3) Is the composition of two translations on a line also a translation?
(4) Is the identity mapping a translation?

3.14 Exercises

1. (a) Given the mapping of W into W with the rule $n \rightarrow n + 2$, what is the range R of this mapping? Is the mapping a translation of W onto R?
   (b) Given the mapping of W into W with the rule $n \rightarrow 2n + 1$, what is the range R of this mapping? Is the mapping a translation of W onto R? Write an argument designed to justify your answer.
2. In this exercise we consider whether the inverse of a translation is also a translation.
   (a) Show that a translation is a one-to-one mapping by an illustration and then using your illustration suggest why any translation is a one-to-one mapping.
   (b) Design an argument (one sentence will do) to show that every translation has an inverse. (We require every translation to be an onto mapping)
   (c) Using an example of a translation (perhaps you might wish to take one with the rule $n \rightarrow n + 5$) show that its inverse is also a translation.
   (d) Construct translations f and g between sets of whole numbers, with the rules $n \rightarrow n + 7$ and $n \rightarrow n - 7$ respectively, so that f and g are inverses of each other. Justify your result.
3. Let f be a translation of W with rule $n \rightarrow n + 3$, and let g be a translation of $\{3, 4, \dots\}$ with rule $n \rightarrow n - 2$.
   (a) What set of whole numbers is W mapped onto under f?
   (b) What set of whole numbers is $\{3, 4, \dots\}$ mapped onto by g?
   (c) Show that the composite map g with f is a translation. What is its rule, domain, and range?
   (d) Show that f with g is a translation. What is its rule, domain and range? Is g with f same translation as f with g?
4. Draw an arrow diagram of the identity mapping of W to W. How is the mapping modified if you reverse the direction of each arrow in your diagram? Does this show that the identity mapping and its inverse are the same mapping?
5. The three points named A, C, B, are such that C is between A and B. Suppose now you apply the translation from W to W with rule $n \rightarrow n + 4$ and use A', C', B', to name the respective images of A, C, B. Tell whether each of the following is true or false and be prepared to justify your answer.

   (a) C' is between A' and B'.
   (b) If C is midway between A and B then C' is midway between A' and B'.
   (c) The distance between A and B is the same as the distance between A' and B'.
   (d) If the translation is applied a second time on A' and B' then the distance between their images will be twice the distance between A and B.
   (e) The inverse translation has the rule $n \rightarrow 4 - n$.
6. Consider the mapping of the dial numbers $\{0, 1, 2, 3, 4\}$ onto itself by the rule $n \rightarrow n + 2$. Can you describe the effect of this mapping on each number in terms of a motion? What kind of motion? Does the mapping of these dial numbers with the rule $n \rightarrow 2n$ also have this motion?
7. Below is a slide rule arrangement for the mapping of W into W with the rule $n \rightarrow 2n$. Notice that the lower ruler is scaled by a unit that is twice as long as the unit of the upper ruler.



   (a) How can you use the arrangement shown above for the mapping of W to W with rule $n \rightarrow n/2$. How is this mapping related to the mapping with rule $n \rightarrow 2n$?
   (b) How would you slide the lower rule to make an arrangement for the mapping of W into W with rule $n \rightarrow 2n + 1$. Make a drawing of this slide rule and check.
   (c) Describe a slide rule arrangement with rule $n \rightarrow 2n - 1$.

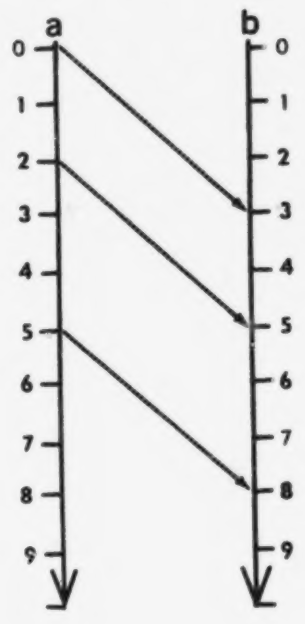
3.15 Mappings from W to W on Parallel lines

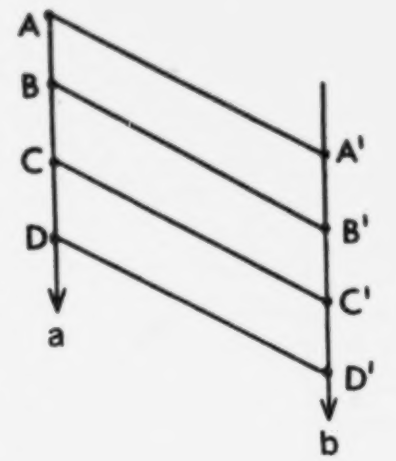
In preceding sections we mapped a set of numbers onto a set of numbers. Sometimes we regarded these numbers as points on a number line. We could do this because each whole number is associated with only one point on a number line and no point represents more than one whole number. So we could think of a number as a point, or a point as a number. In this section we go on to consider mappings from one line to another and we would like to continue thinking of points and numbers interchangeably. But now there will be two points for a given number, one on each line, so we must avoid confusion by clearly specifying the line on which the point lies.

We start with two parallel lines and scale each, using the same unit length. As you see we have named the lines a and b. Let us now consider a mapping, call it f, of W (on a) to W (on b). Let the rule be:

f : n (on a) $\rightarrow$ n + 3 (on b)

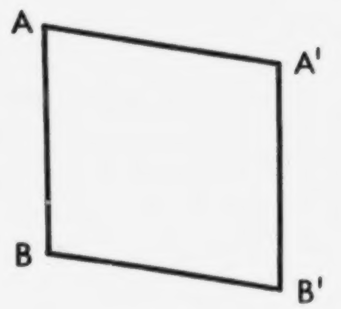
We have shown in the diagram that the image of 0 (on a) is 3 (on b). What is the image of 2 (on a)? Of 4 (on a)? Of 128 (on a)? What is the number whose image is 9 (on b)? Does it occur to you that this mapping can be described by a motion of a definite distance in a definite direction? Then indeed, it is a translation. Does its rule have the form required for a translation?





It would then seem as though the motions of this mapping are in parallel lines. If we consider only two points on a, name them A and B, and their images on b, A' and B' then they determine a figure which is probably familiar to you. It is a parallelogram. The features which make it a parallelogram are the facts that each pair of sides are in parallel lines. The points A, B, B', A' are called its vertices and we name the figure ABB'A', writing vertices in cyclic order, as though we were running bases on a baseball diamond.

Let us take points C and D on a such that A, B, C, D are evenly spaced and let the images of C and D under f be C' and D' respectively. Name another parallelogram whose name starts BC. . . , one whose name starts CD...; AC...; AD... .
Do you think that A', B', C', D' are also evenly spaced? Is the spacing for A, B, C, D the same as the spacing for A', B', C', D'? If so then we can claim that a pair of opposite sides of a parallelogram have the same length. Which pair in the case of parallelogram ABB'A'? But we can think of the parallelogram ABB'A' arising from the translation from the line passing through A and A' to the line passing through B and B'. This means that the other pair of sides AA' and BB' also have the same length. In short in any parallelogram each side is as long as the side opposite; Another way to say this is that a translation preserves distance.



In discussing a parallelogram we have had occassion to refer to lines, sides, which are segments of lines, and lengths of segments. We shall have many more occasions to refer to them again in many other connections. Therefore, it would be convenient to have a simple way to refer to them.

Suppose A and B are two distinct points. Then the line that passes through them will be named $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$. The double arrow suggests that the line extends in both directions. The segment that joins A and B will be named $\overline{AB}$ or $\overline{BA}$. Note that there is no arrow at either end of the bar. The length of $\overline{AB}$ will be named AB (no bar).

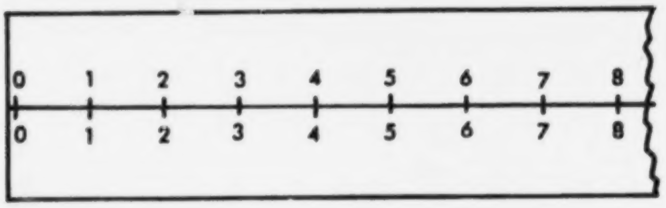
Using this notation, we can say about parallelogram ABB'A'

(1) $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel to each other and $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ are parallel to each other.

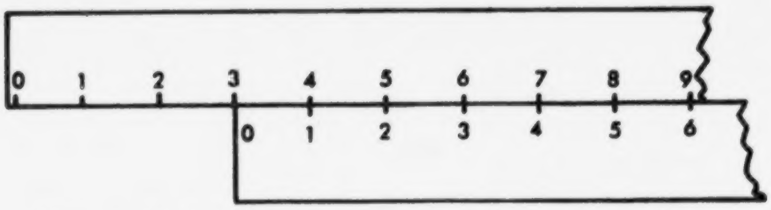
(2) $\overline{AB}$ and $\overline{A'B'}$ have the same length, or AB = A'B'. $\overline{AA'}$ and $\overline{BB'}$ have the same length, or AA' = BB'.

It would be false to say $\overline{AB} = \overline{A'B'}$, because would mean that AB and A'B' are names of the same segment. We can say that AB = A'B' because AB and A'B' are names of the same number.

Let us return to our discussion of translation. We can show a translation of W into W mechanically. We place two rulers, having the same unit, edge to edge so that the same numbers are opposite each other. See below.



Suppose the translation has rule $n \rightarrow n + 3$. Then we slide one of them, say the lower, 3 units to the right. See below.



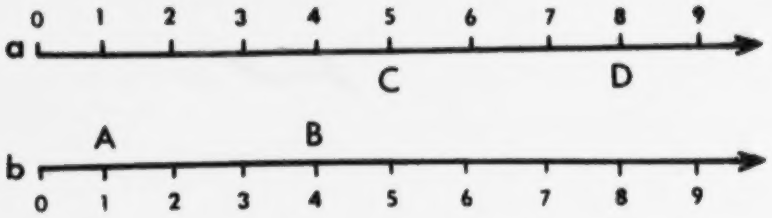
Now to find the image of any number, under this translation, we locate the number in the lower ruler and read off the number opposite it in the upper ruler. For example the image of 4 is 7. Is this not a way of finding the sum of a natural number and 3? How would you modify this procedure to obtain a mechanical way of adding 5? Could it be used also to add $3\frac{1}{2}$? How would you modify to show a subtraction by 3?

This device, in which we slide one ruler's edge along the edge of another, is called a slide rule. You will eventually meet with a variety of slide rules, each stemming out of different kinds of mappings, and each quite useful in our everyday world.

There are many other interesting properties of parallelograms which we shall study and some of these you will meet in the exercises. Also, in the exercises, you will meet with some interesting properties of mappings from W to W on parallel lines. But before we look at these exercises let us note an interesting fact about the figure ABB'A' above. It is this. If AB and A'B' are parallel and AB = A'B', then there is a translation that maps A onto A' and B onto B', assuming AB and A'B' are oriented as in the given figure. Thus if one pair of sides in the figure are in parallel lines and their lengths are equal, then the figure is a parallelogram.

3.16 Exercises

1. Set up two number lines on parallel lines as suggested in the diagram below.



(a) Make an arrow diagram of mapping f from W (on a) to W (on b) with the rule

n (on a) $\rightarrow n + 1$ (on b).

(b) On the same diagram represent mapping g from W (on b) to W (on a) with the rule

n (on b) $\rightarrow n + 3$ (on a).

(c) Using color show the composite of g with f.

(d) Is g with f a translation? If so, what is its rule?

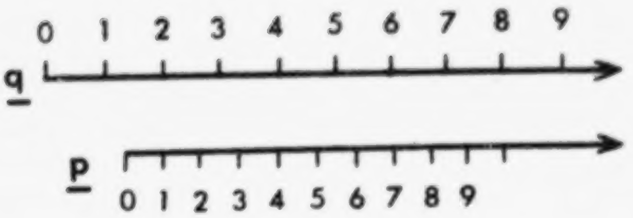
(e) Describe the inverse of f.

(f) Suggest which translation from W (on b) to W (on b) may be regarded as an identity translation. Verify your answer by checking whether the composition of your candidate for the identity translation with f is f.

(g) Find the rule of the mapping which shows that ABDC in the figure of Exercise 1 is a parallelogram.

2. Using two number lines as in Exercise 1, draw an arrow diagram for the mapping from W (on a) to W (on b) with rule n (on a) $\rightarrow 2n$ (on b). Is it a translation?

3. For this exercise copy the two number lines shown below. Note that the lines are scaled with different unit lengths and one is "moved over."



For each of the rules below draw (on separate diagrams) an arrow diagram of the mapping from W (on p) to W (on q).

(a) n (on p) → n + 3 (on q). Is this a translation?

(b) n (on q) → 2n (on p). Is this a translation?

(c) n (on p) → 2n (on q). Is this a translation?

(d) n (on p) → 4 − n. (this is a mapping of a subset of W). Is this a translation? We shall have more to say about the mappings in (b) and (c) in the next section.

(e) Which of these mappings preserve distance?

4. Copy the number lines in Exercise 1 and investigate whether the following statements are true or false for a translation from W (on a) to W on (b). If you wish you may use n (on a) → n + 4 (on b) as the rule of the translation. Let A, B, C be points on a and let their respective images under the influence of the translation be A', B', C'.

(a) If A is to the left of B, then A' is to the left of B'.

(b) If B is between A and C, then C' is between A' and B'.

(c) If A is half way between B and C, then A' is half way between B' and C'.

(d) If C is two thirds of the way from A to B, then C' is one third of the way from B' to A'.

5. In this exercise we use number lines like those in Exercise 1. Let f be the translation from W (on b) to W (on a) such that n (on b) → n + 3 (on a) and let g be the translation from W (on a) to W (on b) such that n → n + 2

(a) Show that g with f is a translation from W (on b) to W (on b). What is the inverse of this composite, expressed with aid of a single rule of the form n (on b) ? (on b)?

(b) Let h be the mapping of W (on a) to W (on a) with the rule n → n + 2. Show that h with f is a translation and also describe the inverse of this composite.

6. (a) Let p and q be two parallel lines, scaled with the same unit. Show with an example that if f is a translation from p to q and g is a translation from q to p, then g with f is a translation from p to p.

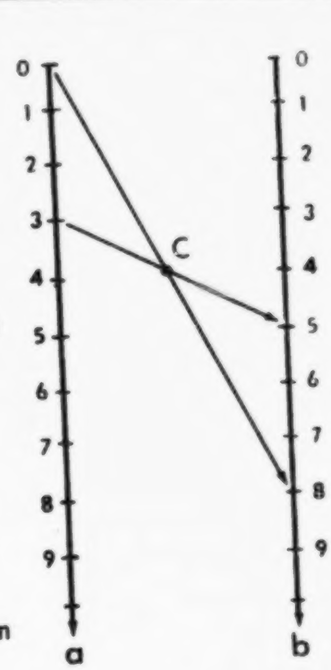
(b) Let f have rule n → n + 2, and let g have rule n → n + 3. Show that g with f is a mapping of W into W that determines the sum of any number and 5.

(c) Let f have rule n → n + 2 and g have rule n → n − 3. Show that g with f is a mapping of N to W that determines the result of first adding 2 and then subtracting 3.

(d) Let f have rule n → n − 2 and g have rule n → n − 3. Show that g with f is a mapping of a subset of W to W that determines the effect of first subtracting 2 and then subtracting 3, and that this effect is to subtract 5. What is its domain?

7. This exercise illustrates an interesting and important mathematical fact. Let p, q, r be three parallel lines, each scaled with the same unit. Show by an example that if f is a translation from p to q, and g is a translation from q to r, then g with f is a translation from p to r, that is the arrows in the arrow diagram of g with f are in parallel lines and have the same lengths.

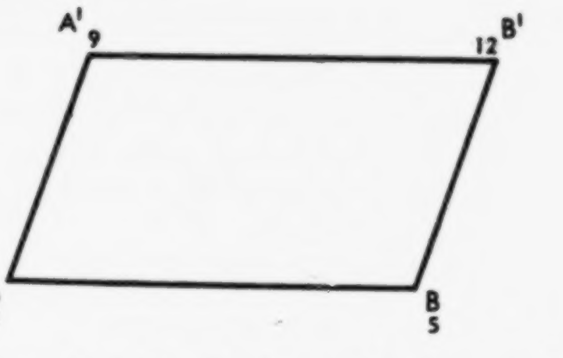
3.17 More on Mappings from W to W on Parallel Lines



There are other interesting mappings between parallel lines. Consider, for instance, two parallel lines, named a and b, scaled by the same unit (shown at the right) and the mapping f, of a subset of W (on a) to W (on b) with the rule n (on a) → 8 − n (on b). Then f : 0 (on a) → 8 (on b), and f : 3 (on a) → 5 (on b). What is the image of 4 (on a)? Of 7 (on a)? What is the domain of f?

Make an arrow diagram like this one at the right and on it show the mapping as it effects {0, 2, 3, 4, 7, 9} on a. What seems to be true about all the arrows? If your drawing is accurate you will see that they all meet in a point. It is the point named C in the diagram. This kind of mapping is called a central projection between parallel lines. (There are other central projections, between intersecting lines, but our interest at the moment is in parallel lines). The point C is called the center of the projection.

Let us consider only two points and their images. You may take any two points on q, numbered less than 9. Name them A and B. Now find the image of A, call it A', and the image of B, call it B'. On your diagram, how many units are there between A and B? How many between A' and B'? Are the lengths of $\overline{AB}$ and $\overline{A'B'}$ equal? Does the projection preserve distance? Thus we can say that ABB'A' is a parallelogram. Now we compare AC with CB'. To make this comparison you can hold the edge of a paper along $\overleftrightarrow{AC}$, mark points on this paper opposite A and C, then move the paper and see if these points match C and B'. Or, you can use your ruler. If you find the distances to be the same then C bisects $\overline{AB'}$, or C is said to be the midpoint of $\overline{AB'}$. Now see if C is also the midpoint of $\overline{BA'}$. As you probably know, AB' and BA' are called diagonals of ABB'A' Our little experiment has shown that the diagonals of ABB'A' bisect each other.



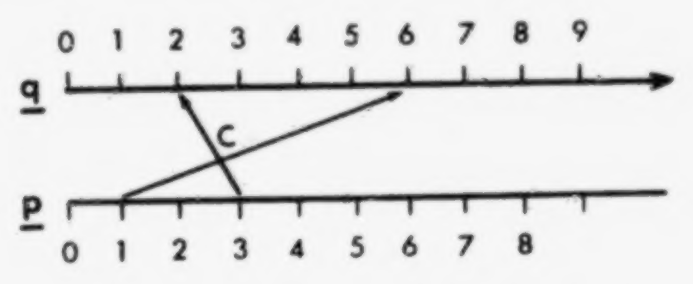
Do you think that the diagonals of any parallelogram bisect each other? Let us see. Suppose ABB'A' is a parallelogram. Then there is a translation from W (on $\overleftrightarrow{AB}$) to W (on $\overleftrightarrow{A'B'}$). Suppose A corresponds to 2 (on $\overleftrightarrow{AB}$) and B corresponds to 5 (on $\overleftrightarrow{AB}$). Let the rule of the translation be $n \longrightarrow n + 7$. Then A' corresponds to 9 (on $\overleftrightarrow{A'B'}$) and B' corresponds to 12 (on $\overleftrightarrow{A'B'}$). We notice that 2 + 12 = 14, that is, the numbers corresponding to opposite vertices have the same sum. Therefore there is a central projection from W (on $\overleftrightarrow{AB}$) to W (on $\overleftrightarrow{A'B'}$) with the rule $n \longrightarrow 14 - n$. This is like the rule of the above central projection. It would seem then that $\overline{AB'}$ and $\overline{A'B}$ should also bisect each other.

A second feature of central projections between parallel lines distinguishes it from translations. Note as you take two numbers in the domain of our central projection in increasing order, then their images are in decreasing order. Is this also true about translation? (We assume that the order of whole numbers on the parallel lines are the same to begin with).

Let us look at a second example of a mapping from W (on p) to W (on q) where p and q are parallel lines shown below with rule:

$$n \longrightarrow 8 - 2n$$

We show its effect on 1 and 3. You are to make your own diagram and answer the questions below.



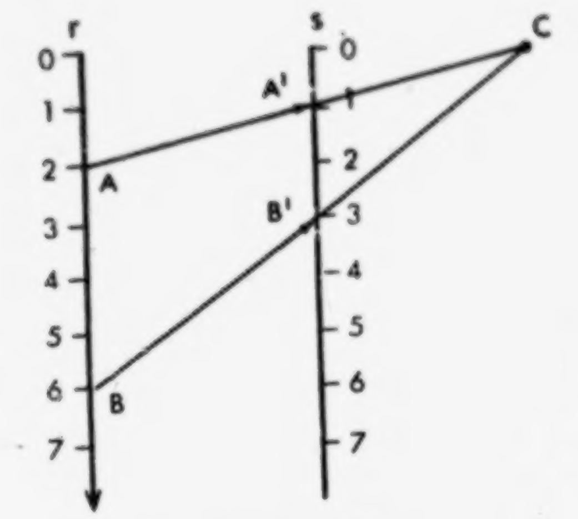
(1) Is this mapping a central projection?
(2) Does the mapping preserve distance?
(3) Does it preserve order?
(4) If A has image A' and C is the center of the projection, compare the lengths of $\overline{AC}$ and $\overline{C'A}$.
(5) If B has image B', compare the lengths of $\overline{AB}$ and $\overline{A'B'}$.

As you can see both central projections reverse order but both do not preserve distance. Thus, a central projection from W to W on parallel lines preserves distance if its rule is of the form.

$$n \longrightarrow a - n$$

where a is some number you know, other than 0. What are the consequences of taking 0 for a?

And now, on the next page, a third example of a mapping f from r to s, two parallel lines with the rule n 1/2m. We show its effect on only two points and you are to investigate its effect on other points on your own diagram, and answer the questions listed below it.



$f: n \longrightarrow \frac{1}{2}n$

(1) Is this a central projection? If so where is the center of the projection?
(2) Does the mapping preserve distance?
(3) Does it preserve order?
(4) If A' is the image of A, compare the length of $\overline{CA}$ with that of $\overline{CA'}$.
(5) If B' is the image of B, compare the length of $\overline{A'B'}$ with that of $\overline{AB}$.

3.18 Exercises

In these exercises we consider mappings from line a to line b, where a and b are parallel lines and scaled with the same unit. Points with the same number are opposite each other.

1. For each rule listed below make an arrow diagram of a mapping from W (on a) to W (on b) and then answer each of the following questions.
   (a) Is the mapping a translation or a central projection, if either?
   (b) Does it preserve distance?
   (c) Does it preserve order?
   (d) If it does not preserve distance, what effect does it have on distance?
   (e) If it is a central projection with center C, and A' is the image of A, compare the length of $\overline{AC}$ with that of $\overline{CA'}$.

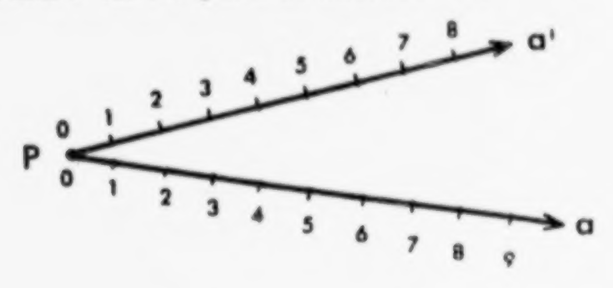
   (1) $n \longrightarrow 5 + n$ (4) $n \longrightarrow 3n$
   (2) $n \longrightarrow 5 - n$ (5) $n \longrightarrow 2n + 1$
   (3) $n \longrightarrow \frac{1}{3}n$ (6) $n \longrightarrow n - 5$

2. Answer the questions listed in Exercise 1 for the mappings with rules listed below without making a diagram.

   (a) $n \longrightarrow 6 - n$ (d) $n \longrightarrow n + 6$
   (b) $n \longrightarrow n - 6$ (e) $n \longrightarrow 3n + 1$
   (c) $n \longrightarrow 4n$ (f) $n \longrightarrow 11 - 3n$

3.20 Parallel Projections

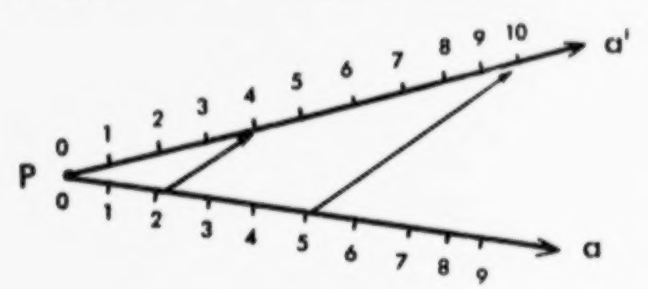
In preceding sections we studied mappings from one of two parallel lines to the other. Let us go on to look at mappings from one of two intersecting lines to the other. As in those sections we continue to scale the two lines with the same unit. Moreover, let us start the scaling at the point of intersection. Then our scaled lines look like this. (Note the names a and a' for the lines, and P for the point of intersection).



We are now ready to map W (on a) into W (on a'). The rule for our first mapping f is to be:

$$n \longrightarrow 2n.$$

We show the effect of f only on 2 and 5.



Make a copy of this mapping diagram and on it draw the arrows that show the image of 1; of 3; of 0. What is the image of 120? On your diagram do you noted that arrows lie in parallel lines? Because this is so, we we have an example of a parallel projection. Consider the following questions for the parallel projection.

(1) Does it preserve order?
(2) Does it preserve distance?
(3) Let A and B be points on a, and let the midpoint of $\overline{AB}$ be C.

(A could be 2, and B could be 6, then C will be 4). Now let A' be the image of A, B' the image of B and C' the image of C. (Then A' is 4, B' is 12, and C' is 8). Is C' the midpoint of $\overline{A'B'}$? This rather complicated situation is described by saying the parallel projection preserves midpoints.

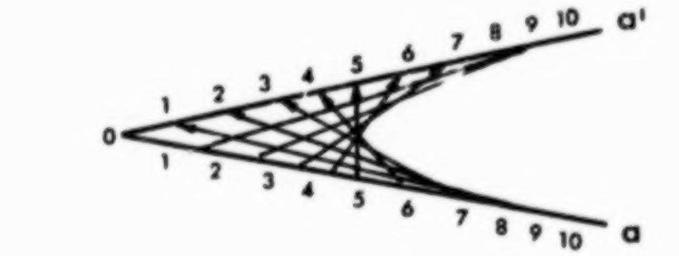
Let us look into this last matter a little more. Suppose D, E, F are points on A corresponding to 1, 3, 6. Then the number of units from D to E is 2; from E to F it is 3. Now let us examine their images D', E' and F'. These correspond to 2, 6, 12. The number of units from D' to E' is 4; from E' to F' it is 6. Do you see

that the position of E between D and F resembles closely the position of E' between D' and F'. The resemblance is due to the fact that 2 and 3 have the same ratio as 4 and 6. In other words, E divides $\overline{DF}$ in the same manner as E' divides $\overline{D'F'}$. When this happens we say that the projection preserves points of division. Do you see that a midpoint is a special case of a point of division?

Look back now and see if translations preserve points of division. Do central projections between parallel lines? A careful study will show that they do indeed. So here we have a property of all three types of mapping that we have studied so far.

Let us look at another mapping between the lines a and a' of out first example. This time we use the rule $n \longrightarrow n + 3$. We leave it to you to decide whether it is a parallel projection. We caution you not to be hasty in arriving at your conclusion.

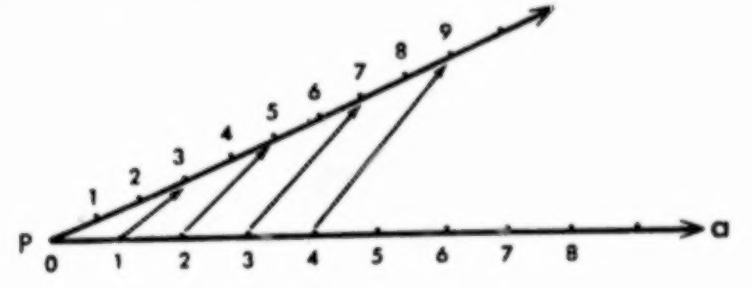
An interesting result emerges from the mapping from N on a to N on a', for the rule $n \longrightarrow 10 - n$.



As you see, the map suggests a curve. It is called a parabola. It has many interesting and useful properties which we are not ready to study, but will be before long.

We end this section with a mapping from line a to line a'. Let the rule of the mapping be

$$n \longrightarrow 2n + 1$$



You may find it difficult to decide whether this is a parallel projection or not. See if it preserves midpoints. See if it preserves any point of division. You might consider for instance, the points on a at 1, 3, 5 and their images. The evidence seems to be in favor of the guess that this mapping is a parallel projection. To some, the arrow lines may appear parallel. But before you decide, consider also the arrow line which tells the image of 0. What is your decision now?

We state what seems to follow from our study of a mapping from W on a to W on a', where a and a' are scaled by the same unit and have the zero point in common. Such a mapping is a parallel projection if it has a rule of the form.

$$n \longrightarrow dn, \text{ (d any fixed natural number)}$$

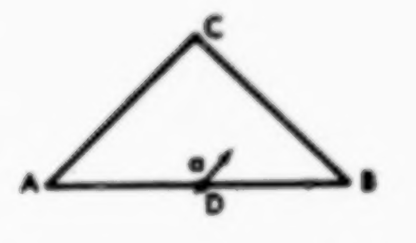
3.21 Exercises

In the following exercises a and a' are names of lines and all mappings are from W on a to W on a'. Assume that a and a' are scaled with the same unit, unless otherwise specified. Also assume, when a and a' intersect, that the scaling starts from the point of intersection. If the lines are parallel assume that the same numbers on each appear opposite each other like the partners in a Virginia reel.

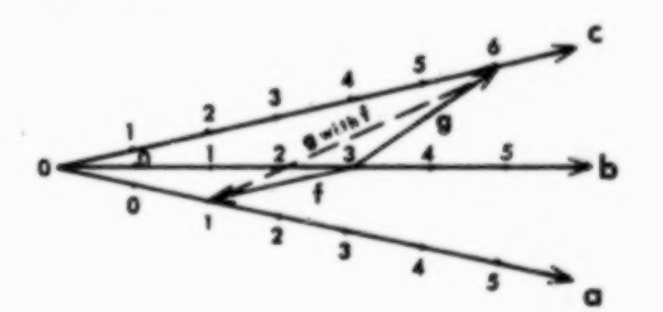
1. Let a and a' intersect. For each of the following rules make an arrow diagram.

   (a) $n \longrightarrow 3n$ (d) $n \longrightarrow n + 2$
   (b) $n \longrightarrow 3n + 1$ (e) $n \longrightarrow 8 - n$
   (c) $n \longrightarrow 3n - 1$ (f) $n \longrightarrow n$

2. Which of the mappings in Exercise 1 are parallel projections? Which preserve order? Which preserve distance? Which preserve midpoints? Which preserve division points?
3. Without making a diagram describe the kind of mapping from a to a' that goes with each of the following conditions:
   (a) $n \longrightarrow n + 2$ and a is parallel to a'.
   (b) $n \longrightarrow 2n$ and a is parallel to a'.
   (c) $n \longrightarrow 2n$ and a intersects a'.
   (d) $n \longrightarrow 8 - n$ and a is parallel to a'.
   (e) $n \longrightarrow 2n + 1$ and a is parallel to a'.
4. (a) Suppose a mapping preserves distance. Must it also preserve midpoints? Write an argument to support your answer.
   (b) Suppose a mapping preserves midpoints. Must it also preserve distance? Write an argument to support your answer.
5. (a) Suppose a mapping preserves order. Must it also preserve distance?
   (b) Suppose a mapping preserves distance? Must it also preserve order?



6. Suppose in the diagram above, D is the midpoint of $\overline{AB}$ and line a passing through D is parallel to $\overleftrightarrow{AC}$. Must the line a pass through the midpoint of $\overline{BC}$? Write an argument to support your answer.
7. Consider the three properties listed below for mappings
   (1) It preserves order.
   (2) It preserves distance.
   (3) It preserves points of division.
   Describe a mapping that has the following properties
   (a) (1), (2), and (3)
   (b) (1) and (3), but not (2)
   (c) (2) and (3) but not (1)
   (d) only (3)



8. Let lines a, b, and c meet in one point, as shown above and let each be scaled by the same unit, starting at the point of intersection. Let f be the mapping with rule n (on a) ⟶ 3n (on b), and g the mapping with rule n (on b) ⟶ 2n (on c).
   (a) Describe the inverse of f, if it has one.
   (b) Describe the composition of g with f and find its rule.
   (c) Show that the composition of g with f is a parallel projection.

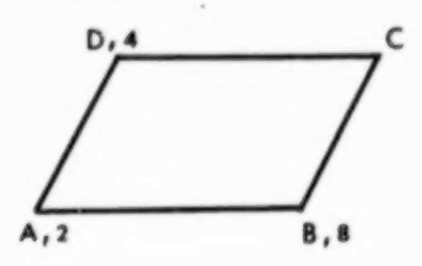
3.22 Summary

1. The process of mapping involves two sets and the assignment to a member of one set exactly one image taken from the other set. The set of objects that have images is called the domain of the mapping; the set of images is called the range of the mapping. In this chapter many mappings had rules of the form $n \longrightarrow an + b$, where a and b are fixed numbers in each mapping and n takes on whole number values.
2. In any mapping each member of the domain is assigned exactly one image. In a one-to-one mapping each image serves only once as an image. When a mapping is one-to-one and onto it has an inverse. It is formed by interchanging the roles of the domain and range of the original mapping and "inverting" the rule.
3. If the range of mapping f is the domain of mapping g, we can form the composite of the two. If f is followed by g the composite is called g with f. The composite of a mapping and its inverse (if it has one), in either order, is an identity mapping.
4. An onto mapping from W to W on a line with rule $n \longrightarrow n + a$, or $n \longrightarrow n - a$, where a is a fixed number for each mapping, is a translation. It may be regarded as a motion through distance a in the direction of increasing numbers for the first rule, and in the opposite direction for the second rule.
5. We also studied mappings from W as one of two parallel lines to W on the other; the lines are scaled with the same unit and numbers on these lines increase in the same direction. If the mapping has a rule of the form $n \longrightarrow n + a$, or $n \longrightarrow n - a$ it is a translation; If the rule has the form $n \longrightarrow an$, then the mapping is a central projection. Central projections preserve division points.
6. A parallel projection from W on one of two parallel lines to W on another is a translation, if the lines are scaled in the same direction with the same unit. A parallel projection from W on one of two intersecting lines to W on another preserves order and points of division, if the lines are scaled from the point of intersection with the same unit.
7. Each side of a parallelogram has the same length as the opposite sides and its diagonals bisect each other.

3.23. Review Exercises

1. Let f be a mapping on a line from W to W with rule $n \longrightarrow n + 3$. Let g be a mapping on this line from W to W with the rule $n \longrightarrow 2n$.
   (a) Make an arrow diagram showing f, g and the composite g with f.
   (b) Find the rule of g with f expressed in the form $n \longrightarrow ?$
   (c) Find the image of 2 under the composite g with f; also find the image of 2 under f with g. Are the two images the same?
   (d) Find the number x such that the composite g with f maps x onto 24.

2. Let a and b be parallel lines, scaled with the same unit and such that numbers increase in the same direction.
   (a) Make an arrow diagram of the mapping f from W (on a) to W (on b) with rule $n \longrightarrow n + 4$.
   (b) What kind of mapping is f?
   (c) Is f a one-to-one mapping? Support your answer with a brief discussion.
   (d) Describe the domain, range, and rule of the inverse of f.
   (e) Show by a composition of f with its inverse that the composite is an identity mapping.
3. Draw an arrow diagram for the mapping from W on line a to W on line a' for each of the following conditions.
   (a) a is parallel to a'; the lines are scaled with the same unit with same direction; the rule is n (on a) ⟶ 2n − 1 (on a').
   (b) a is parallel to a'; the unit scale on a is twice as long as the unit scale on a'; the lines are scaled in the same direction with zero points opposite each other; the rule is n (on a) ⟶ 2n (on a').
   (c) a intersects a' at point A; the lines are scaled with the same unit from A; the rule is n (on a) ⟶ 2n (on a').
   (d) a intersects a' at point A; the unit on a is half as long as the unit on a'; the lines are scaled from A; the rule is n (on a) ⟶ $\frac{1}{2}n + 1$ (on a').
   (e) a is parallel to a'; the lines are scaled with the same unit in opposite directions; the rule is n (on a) ⟶ n+2 (on a').
   (f) a is parallel to a'; the lines are scaled in opposite directions; the scale unit of a is twice as long as the scale unit of a'; the rule is n (on a) ⟶ 2n − 1 (on a').
4. For the mappings in Exercise 3 tell:
   (a) which preserve order
   (b) which preserve distance
   (c) which preserve division points
5. Figure ABCD is a parrallelogram in which the numbers appearing near each vertex tell its position on a number line.



   (a) For what mapping is D the image of A and C the image of B?
   (b) For what mapping is C the image of A and D the image of B?
6. Make an arrow diagram for each of the following mappings where Z_n is the set of n dial numbers and the operations are the dial operations, + and · .
   (a) From Z_5 to Z_5 with rule $n \longrightarrow 2n + 1$.
   (b) From Z_4 to Z_4 with rule $n \longrightarrow 2n$.
   (c) Which of the mappings in (a) and (b) are one-to-one? For these find the rule of the inverse mapping.

CHAPTER 4

The Integers

4.0 Introduction

In chapters 1 and 2 you have studied various operational systems. Two of these were $(Z_5, +)$ and $(W, +)$. In each system, open sentences of the form $a + x = b$ were studied. Some sample open sentences, for these operational systems are:

(1) $3 + x = 2$, $(Z_5, +)$
(2) $1 + x = 4$, $(Z_5, +)$
(3) $12 + x = 18$, $(W, +)$
(4) $11 + x = 6$, $(W, +)$

Which of these open sentences have solutions in the given operational system? Do you see that (1), (2), and (3) have solutions, but (4) does not?

Let us look a little more closely at example (1). In (1) the sentence $3 + x = 2$ corresponds to the sentence $x = 2 - 3$. Thus, the fact that $3 + x = 2$ has a solution in $(Z_5, +)$ is the same as saying that "$2 - 3$" names an element in Z_5.

What about example (4)? To find a solution for (4), we examine the corresponding subtraction sentence $x = 6 - 11$. But "$6 - 11$" does not name an element of W. You will recall that subtraction is not an "always possible" process in $(W, +)$. That is, subtraction is not an operation on the set of whole numbers.

This situation should lead you to feel some dissatisfaction with the operational system $(W, +)$. We cannot solve every open sentence of the form $a + x = b$, where a and b are whole numbers. That this difficulty is more than theoretical is illustrated by the following problem.

> The temperature at 8:00 P.M. is 43°. The weather bureau predicts that the low temperature during the night will be 29°. How many degrees of temperature change must occur?

For this problem a corresponding mathematical sentence is $43 + x = 29$. Now this mathematical sentence has no solution, but we know that the number of degrees of temperature change is 14°. However, this is not enough information, for a temperature change of 14° could be upward to 57° or downward to 29°, as required. A complete answer requires both a number and some additional information. In this case, the temperature must change 14° downward.

This chapter will be devoted to developing an operational system which will always provide solutions to equations of the form $a + x = b$, where a and b are elements of the system, and which will preserve all the nice properties of $(W, +)$.

4.1 Directed Numbers

While Fred and Jim were camping last summer, they spent much of their time exploring the countryside around their campsite. One day they returned to camp excitedly waving a ragged scrap of paper they had found on the edge of a field. It looked like this:

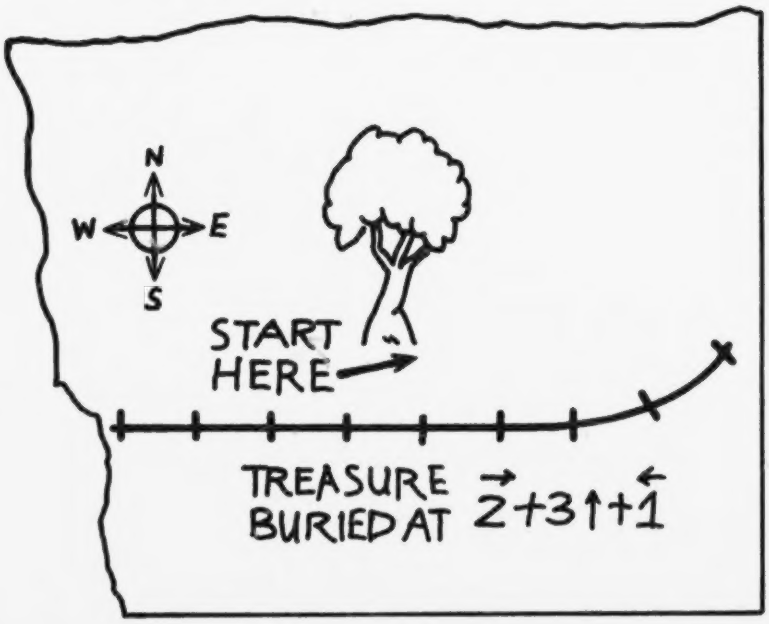


Figure 4.1

Fred and Jim realized that this was a map showing how to locate a buried treasure. However, what on earth did the maker of the map mean by "2 + 3 + 1"?

"Where's Jack" they shouted. (Jack was of course, the math whiz in the group) "This is the first time I have seen a two with an arrow over it or a three with an arrow next to it," said Jack, "but I guess this must have something to do with the compass directions on the map." At this point, Jim became very excited and shouted, "Sure, that's it. The arrows tell us in which direction to go! The first arrow means walk east, the second arrow means walk north, and the third arrow means walk west."

"That sounds reasonable," said Fred, "But how far do we walk each way?" For a while, there was silence - they were stumped. The numbers obviously had something to do with the problem, but what did they stand for?

Jack, with his logical training and keen powers of observation was the first to see the clue. "Look at the map, fellows," he shouted, "There is one item we have ignored so far - the railroad track!"

"Well what about it?" they asked. "Don't you see, on the map the ties are evenly spaced. Let's go back to the railroad and measure the spacing of the ties." "We don't have to," said Jim. "I walked along the track yesterday. The distance between ties fits my stride perfectly, and I happen to know that my stride is just about one yard."

"Fine," said Jack, "then hand me that map and I'll show you how to locate 'the treasure'."

He marked the map as shown in Figure 4.2.

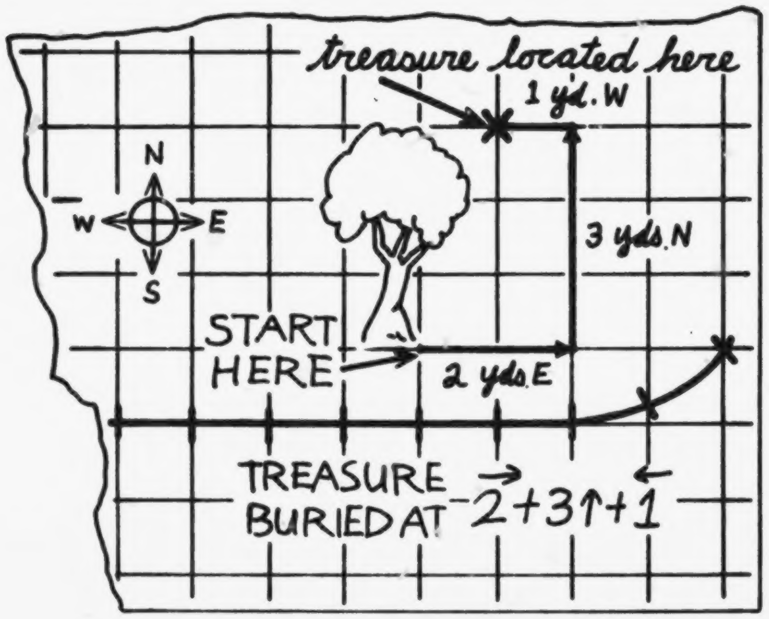

treasure located here
1 yd. W
N
W
E
S
3 yds. N
START HERE
2 yds. E
TREASURE BURIED AT $\overrightarrow{2}+3\uparrow+\overleftarrow{1}$

Figure 4.2

Then they all ran back to the field and Jim paced off the path shown on the map, starting from the foot of the tree. That is where they dug.

Notice, that Jack interpreted the instruction "$\overrightarrow{2} + 3\uparrow + \overleftarrow{1}$" in a very special way. (For convenience, let us agree to read this expression as follows: "Two right, plus three up, plus one left.") In the first place, he realized that the symbols $\overrightarrow{2}$ ("Two right") $3\uparrow$ ("Three up") and $\overleftarrow{1}$ ("One left") did not stand for ordinary whole numbers, but rather a new kind of "directed number," which told how far to walk and in which direction. Actually the symbol $\overrightarrow{2}$ can be interpreted as naming a mapping which takes each point to a new point 2 units east of its original position. Similarly, the symbol "$3\uparrow$" represents a mapping which takes each point to a new point 3 units directly north of its original position.

In the second place, Jack realized that the plus signs did not mean ordinary addition. The expression "$\overrightarrow{2} + 3\uparrow + \overleftarrow{1}$" simply indicated that a motion of two yards towards the east was to be followed by another motion of three yards towards the north and this in turn followed by a third motion of one yard towards the west. The "+" sign as used here can therefore be interpreted as naming a composition operation i.e. a composition of mappings. For example $\overrightarrow{2} + 3\uparrow$ can be interpreted as a single mapping formed by combining (composing) the mapping $\overrightarrow{2}$ and the mapping $3\uparrow$.

The composition operation "followed by" is conveniently represented by a plus sign "+" because as we shall see it indeed resembles in many important ways the ordinary addition operation for whole numbers. In fact it will be very convenient to call this composition operation "addition", provided it is clearly understood, that addition of directed numbers (mappings) is a new operation. We shall see that it is a more general kind of addition which, in a certain sense, includes ordinary addition of whole numbers as a special case.

Returning to Figure 4.2, what is a shorter instruction for locating the treasure? Your answer could be either

$\overrightarrow{1} + 3\uparrow$ (1 yard east followed by 3 yards north

or

$3\uparrow + \overrightarrow{1}$ (3 yards north followed by 1 yard east

Since each of these instructions gives the same result (same mapping) as the original instruction on the treasure map, we call these instructions equal:

$$\overrightarrow{2} + 3\uparrow + \overleftarrow{1} = \overrightarrow{1} + 3\uparrow$$
$$\overrightarrow{2} + 3\uparrow + \overleftarrow{1} = 3\uparrow + \overrightarrow{1}$$
$$\overrightarrow{1} + 3\uparrow = 3\uparrow + \overrightarrow{1}$$

Another instruction, equal to each of these, is $\overrightarrow{2} + 5\uparrow + \overleftarrow{1} + 2\downarrow$ (Read "$2\downarrow$" as "two down": what does "$2\downarrow$" mean here?). Find some others. It will be convenient to talk about the "length" of an instruction. We shall say that the length of the instruction "$\overrightarrow{7} + 5\uparrow + \overleftarrow{4}$" is 3 because this instruction has 3 addends. The length of "$\overrightarrow{7}$" is 1. In general, the length of an instruction is the number of addends for the instruction.

4.2 Exercises

1. In Figure 4.3, name the point located by the mapping $\overrightarrow{3} + 5\uparrow + \overleftarrow{4}$ if we start at
   (a) Point A
   (b) Point B
   (c) Point C

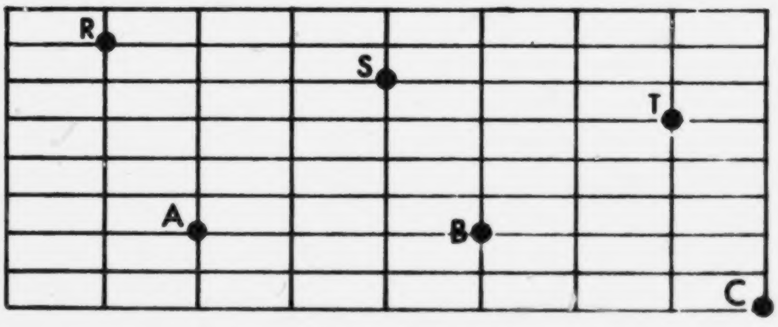


Figure 4.3

2. Give two shorter instructions (of length 2) for the mapping

$$\overrightarrow{3} + 5\uparrow + \overleftarrow{4}$$

3. Referring to Figure 4.3, state a simple instruction mapping which moves

(a) A to S
(b) B to T
(c) A to B
(d) B to C
(e) R to S
(f) S to R
(g) T to A
(h) A to T

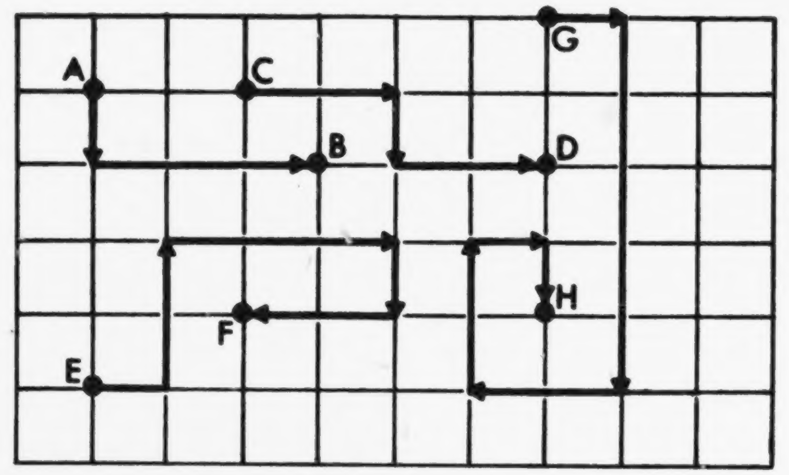


Figure 4.4

4. Referring to Figure 4.4
   (a) State a combination of directed numbers (composition of mappings) corresponding to each path shown.
   (b) If possible, state a shorter instruction for each of the combined mappings in (a). (An instruction of smaller length.)
5. (a) Referring to Figure 4.3, give an instruction for a mapping that takes you from A to T
   1. Through point R
   2. Through point S
   3. Through point B
   4. Through point C

   (b) For each instruction in (a), draw a diagram to show the actual path from A to T.
6. Find the solution for each equation below using an instruction whose length is as short as possible.

Example: $4\downarrow + \overrightarrow{3} + \overrightarrow{2} = \boxed{\quad\quad}$, Solution= $4\downarrow + \overrightarrow{5}$

Exercise Set 1 (Continued)

(a) $4\downarrow + 6\uparrow = \square$ Solution =
(b) $4\uparrow + 6\downarrow = \square$ Solution =
(c) $\overrightarrow{4} + \overrightarrow{6} = \square$ Solution =
(d) $\overrightarrow{4} + \overleftarrow{6} = \square$ Solution =
(e) $\overleftarrow{4} + \overleftarrow{6} = \square$ Solution =
(f) $\overleftarrow{4} + \overrightarrow{6} = \square$ Solution =
(g) $\overrightarrow{6} + \overleftarrow{4} = \square$ Solution =
(h) $\overrightarrow{6} + \square = \overrightarrow{9}$ Solution =
(i) $\overrightarrow{6} + \square = \overrightarrow{4}$ Solution =
(j) $\overrightarrow{6} + \square = 0$ Solution =

Note: $\overrightarrow{0} = \overleftarrow{0} = 0\downarrow = 0\uparrow$ (Each of these is a ''zero'' mapping i.e. A motion of 0 units in the direction indicated. In fact, it is our identity mapping,)

(k) $\overrightarrow{6} + \square = \overrightarrow{6}$ Solution =
(l) $\overrightarrow{6} + \square = \overleftarrow{6}$ Solution =
(m) $\overrightarrow{2} + 3\uparrow + \square = \overrightarrow{3} + 5\uparrow$ Solution =
(n) $\square + \overrightarrow{6} = 2\downarrow$ Solution =
(o) $\square + \overrightarrow{6} = 2\uparrow$ Solution =
(p) $\square + \square = \overrightarrow{10}$ Solution =
(q) $\overrightarrow{2} + 3\downarrow + \square = \overrightarrow{0}$ Solution =
(r) $\overrightarrow{2} + 3\downarrow + \square = \overleftarrow{5}$ Solution =
(s) $\overrightarrow{2} + \square + 3\downarrow = \overleftarrow{5} + 3\downarrow$ Solution =

7. Answer either TRUE or FALSE for each of the following:

(a) $\overrightarrow{3} + 2\downarrow = 3\downarrow + \overrightarrow{2}$
(b) $\overrightarrow{3} + 2\downarrow = \overrightarrow{2} + 3\downarrow$
(c) $\overrightarrow{3} + 2\downarrow = \overrightarrow{2} + 3\downarrow$
(d) $\overrightarrow{3} + 2\downarrow = \overrightarrow{5} + 2\downarrow + \overleftarrow{2}$
(e) $\overrightarrow{3} + \overleftarrow{3} = 6\downarrow$
(f) $(\overrightarrow{3} + \overleftarrow{6}) + \overrightarrow{2} = \overrightarrow{3} + (\overleftarrow{6} + \overrightarrow{2})$
(g) If $\overrightarrow{2} + \square + 5\downarrow = \overleftarrow{3} + 5\downarrow$
Then $\overrightarrow{2} + \square = \overleftarrow{3}$
(h) If $\overrightarrow{2} + \square + 5\downarrow = \overrightarrow{2} + \overleftarrow{6}$
Then $\square + 5\downarrow = \overleftarrow{6}$

8. Find the solution set for each of the following equations writing ''simplest'' names, where we now permit fractional directed numbers.

(a) $\overrightarrow{\frac{3}{4}} + \overrightarrow{\frac{1}{2}} = \square$
(b) $\overrightarrow{\frac{3}{4}} + \overleftarrow{\frac{1}{2}} = \square$
(c) $\overleftarrow{\frac{3}{6}} + \overleftarrow{\frac{1}{2}} = \square$
(d) $\overrightarrow{2.6} + \overrightarrow{7.4} = \square$
(e) $\overrightarrow{2.6} + \overleftarrow{7.6} = \square$
(f) $\overrightarrow{2.6} + \square = \overrightarrow{7.4}$
(g) $\overrightarrow{2.6} + \square = \overleftarrow{7.4}$

9. As in the case of whole numbers we can introduce subtraction for directed numbers. For example:

$$5 - 2 = 3 \quad \text{because } 5 = 3 + 2$$

In general a − b = c if and only if a = c + b
For directed numbers we have $\overrightarrow{5} - \overrightarrow{2} = \overrightarrow{3}$ because
$\overrightarrow{5} = \overrightarrow{3} + \overrightarrow{2}$

For each subtraction sentence we have a corresponding addition sentence. Thus for $\overrightarrow{5} - \overrightarrow{2} = \square$ we have the equivalent addition sentence $\overrightarrow{5} = \square + \overrightarrow{2}$. Find an equivalent addition sentence for each of the following subtraction sentences and then express its solution in terms of fewest directed numbers.

(a) $\overrightarrow{4} - \overrightarrow{3} = \square$
(b) $\overleftarrow{6} - \overleftarrow{3} = \square$
(c) $\overrightarrow{5} - \overleftarrow{3} = \square$
(d) $\overrightarrow{5} - 3\uparrow = \square$
(e) $(\overrightarrow{5} + \overrightarrow{2}) - \overleftarrow{3} = \square$
(f) $(\overrightarrow{7} + \overrightarrow{6}) - \overrightarrow{2} = \square$
(g) $\overrightarrow{7} - (\overrightarrow{6} + \overrightarrow{2}) = \square$
(h) $\overrightarrow{7.5} - \overrightarrow{2.7} = \square$
(i) $\overrightarrow{7.5} - \overleftarrow{2.7} = \square$

4.3 Addition Properties for Directed Numbers

In the last section we said that addition of directed numbers resembled ordinary addition of whole numbers in many important ways. Let us look at some of these resemblances in more detail.

We saw that the original instruction for locating the treasure could be replaced by either of two simpler instructions namely 1 + 3 or 3 + 1. Each of these denotes a composition of two mappings (instead of three mappings as in the original instruction). Let us now observe the simple but very fundamental geometrical fact that the treasure can be reached by following either of two paths, starting from the tree:

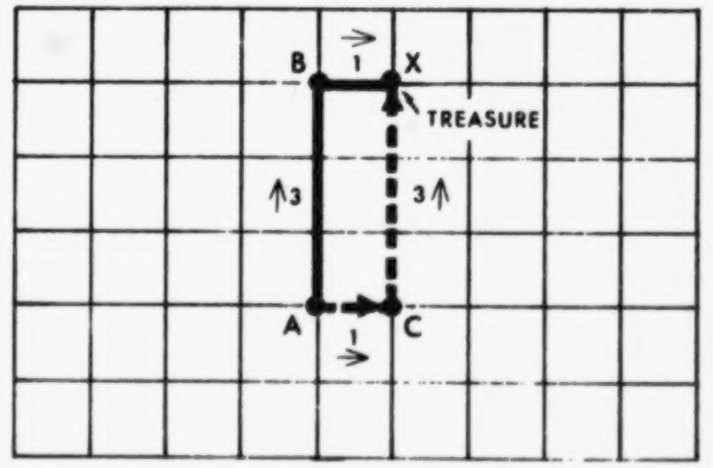


Figure 4.5

The addition indicated by $\overrightarrow{1} + 3\uparrow$ is a composite mapping which take us from point A to point X along a path through point C in Figure 4.5. The addition indicated by 3 + 1 is a composite mapping that takes us from point A to point X along a path through point B. Although the paths are different, they lead to the same spot and are equally good for locating the treasure. Therefore we write

$$\overrightarrow{1} + 3\uparrow = 3\uparrow + \overrightarrow{1}$$

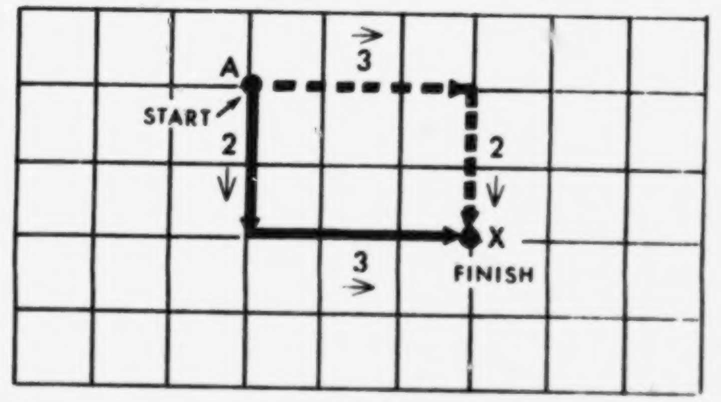
This equality suggests the commutative property which we already know holds for other kinds of addition such as clock addition and ordinary addition of whole numbers. Recall, for example that the commutative law for addition of whole numbers may be stated as follows

For all whole numbers a and b,
a + b = b + a

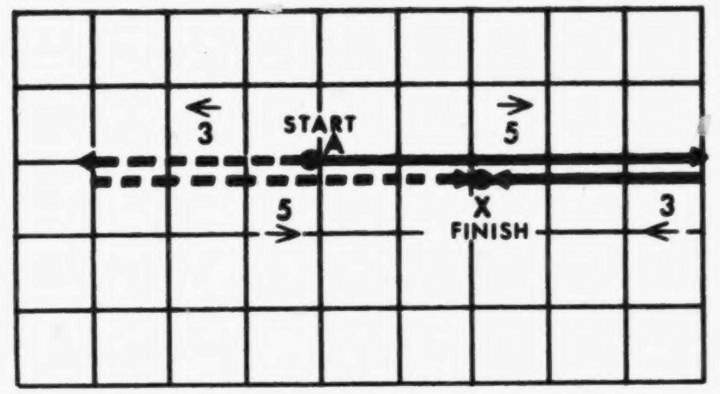
Does a similar law apply to addition of directed numbers? You may have observed that there are mappings for which composition is not a commutative operation. Our addition operation for directed numbers is actually a composition of mappings (translations in certain directions). We would like to know whether addition of directed numbers is commutative.

Let us first test the commutative property on a few more pairs of directed numbers.

Example: Does $2\downarrow + \overrightarrow{3} = \overrightarrow{3} + 2\downarrow$?



Example: Does $\overrightarrow{5} + \overleftarrow{3} = \overleftarrow{3} + \overrightarrow{5}$?



Of course we can continue testing other special cases (make up a few more examples for yourself and try them out). Each of these special cases strengthens our belief that additon of directed numbers is indeed commutative. But special cases do not prove a general rule. To prove a general rule we need a general argument, that is a line of reasoning which applies to all possible special cases.

Although we are not in a position to give a general proof at this stage, we can present a fairly convincing argument to show that addition of directed numbers is commutative. We do this by observing that each directed number corresponds to a translation either along a line running north and south, or along a line running east and west. Now suppose a and b are directed numbers. Then the sum a + b is a composite mapping which takes any point A (''start'') to a new point X (''finish'') by a path which proceeds along two sides of a rectangle. (See Figure 4.6).

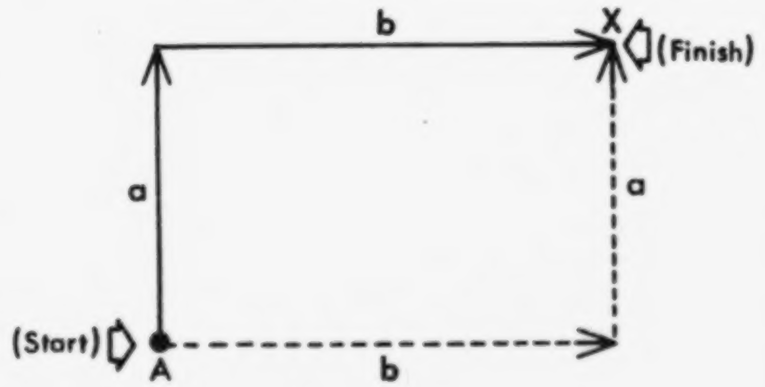


Figure 4.6

The sum b + a is also a composite mapping which takes the starting point A to the very same final point X except that the path now consists of the other two sides of the same rectangle. The paths may be different but whenever they start at the same point they finish at the same point. Thus the composite mappings a + b and b + a produce exactly the same effect on every point in their domain, i.e. ''a + b'' and ''b + a'' actually represent the same mapping. Addition of directed numbers therefore is commutative:

For all directed numbers a and b,
a + b = b + a

Another fundamental property of addition of whole numbers is associativity:

For all whole numbers a, b, c,
(a + b) + c = a + (b + c)

For example (7 + 3) + 8 = 7 + (3 + 8). (Notice that each adds up to 18). Does the associative property also hold for addition of directed numbers? Once again, let us check a special case.

Example: Does $(\overrightarrow{7} + \overleftarrow{3}) + \overleftarrow{8} = \overrightarrow{7} + (\overleftarrow{3} + \overleftarrow{8})$?

Let us find a simpler expression for each side:
$\overrightarrow{7} + \overleftarrow{3} = \overrightarrow{4}$

Therefore $(\overrightarrow{7} + \overleftarrow{3}) + \overleftarrow{8} = \overrightarrow{4} + \overleftarrow{8} = \overleftarrow{4}$

Also $\overleftarrow{3} + \overleftarrow{8} = \overleftarrow{11}$ Same

Hence $\overrightarrow{7} + (\overleftarrow{3} + \overleftarrow{8}) = \overrightarrow{7} + \overleftarrow{11} = \overleftarrow{4}$

This proves that $(\overrightarrow{7} + \overleftarrow{3}) + \overleftarrow{8} = \overrightarrow{7} + (\overleftarrow{3} + \overleftarrow{8})$. (Make up a few more examples for yourself and try them out).

As we pointed out before, special cases do not prove a general rule. We need a general argument that will apply to all cases. This time we are most fortunate because a very neat and highly satisfactory logical argument can be given to show that the associativity property holds, precisely because addition of directed numbers is a composition of mappings.

Suppose a, b and c are any directed numbers and that P is any starting point. Suppose further that

a maps point P onto point Q
b maps point Q onto point R
c maps point R onto point S

We might picture these mappings as shown in Figure 4.7.

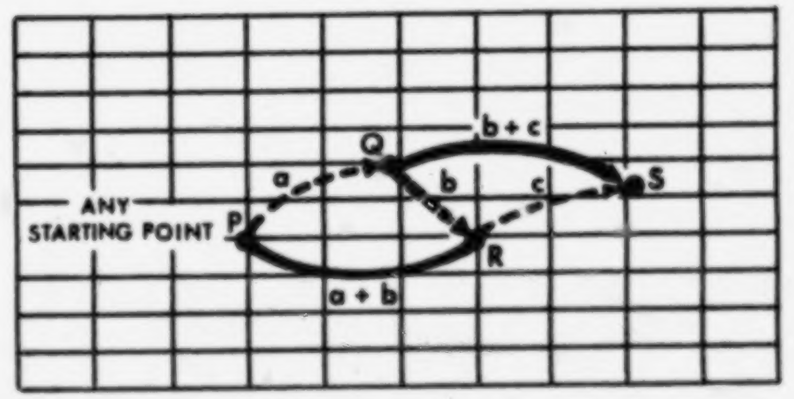


Figure 4.7

The sum $a + b$ is a composite mapping. It maps point P onto point R. Furthermore c maps point R onto point S. Hence, it follows that

$(a + b) + c$ maps point P onto point S.

Now let us examine the effect of $a + (b + c)$ on point P. We know that a maps point P onto point Q. The sum $b + c$ is a composite mapping which maps point Q onto point S. By combining a with $b + c$ we see that

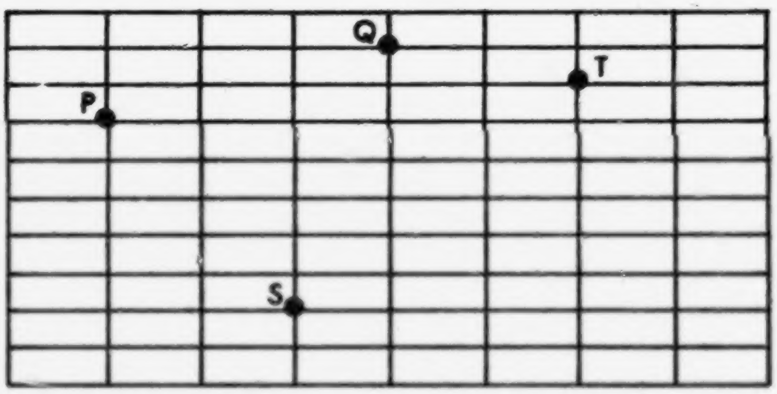
$a + (b + c)$ maps point P onto point S

In other words $a + (b + c)$ produces the same effect on point P as $(a + b) + c$. But remember that P represented any starting point. We have therefore proved that the mapping $(a + b) + c$ produces the same effect on any starting point as does the mapping $a + (b + c)$. This means that addition of directed numbers is indeed associative.

For all directed numbers a, b, c,
$(a + b) + c = a + (b + c)$

In fact, the above argument, proving associativity of addition of directed numbers, is the same argument that is used to prove associativity of composition of mappings in general.

4.4 Exercises



1. Refer to the figure shown. To points P and S apply each of the following mappings:

   $\overrightarrow{3} + 2\uparrow$ and $2\uparrow + \overrightarrow{3}$

   Draw the path for each instruction. (Use your own paper)

2. Write the corresponding addition sentence for each of the following subtraction sentences

   $\overrightarrow{7} - \overrightarrow{3} = \square$
   $\overrightarrow{3} - \overrightarrow{7} = \square$

   Does $\overrightarrow{7} - \overrightarrow{3} = \overrightarrow{3} - \overrightarrow{7}$?

   How do the maps $\overrightarrow{7} - \overrightarrow{3}$ and $\overrightarrow{3} - \overrightarrow{7}$ compare?

   Is subtraction commutative?

3. Name at least one other operation that is not commutative.

4. Write the shortest possible instruction for each of the following mappings (an instruction with fewest directed nos.)

   (a) $(\overrightarrow{7} + \overrightarrow{5}) + \overleftarrow{2}$ and $\overrightarrow{7} + (\overrightarrow{5} + \overleftarrow{2})$
   (b) $(\overrightarrow{7} + \overleftarrow{5}) + \overleftarrow{2}$ and $\overrightarrow{7} + (\overleftarrow{5} + \overleftarrow{2})$
   (c) $(\overrightarrow{7} + \overrightarrow{3}) + \overleftarrow{10}$ and $7 + (3 + 10)$
   (d) $\overrightarrow{7} - \overrightarrow{3}$ and $\overrightarrow{7} + \overleftarrow{3}$
   (e) $\overrightarrow{3} - \overrightarrow{7}$ and $\overrightarrow{3} + \overleftarrow{7}$
   (f) $(\overrightarrow{3} - \overrightarrow{7}) - \overrightarrow{3}$ and $\overrightarrow{3} - (\overrightarrow{7} - \overrightarrow{3})$
   (g) $(\overrightarrow{9} - \overrightarrow{3}) - \overrightarrow{5}$ and $\overrightarrow{9} - (\overrightarrow{3} - \overrightarrow{5})$. Compare this with $(9 - 3) - 5$ and $9 - (3 - 5)$

5. Is subtraction of whole numbers associative?

6. Is subtraction of directed numbers associative?

7. Name at least one other operation that is not associative.

8. Solve

   (a) $a + \overrightarrow{7} = \overleftarrow{10}$
   (b) $\overleftarrow{6} + b = \overrightarrow{6}$
   (c) $\overrightarrow{5} = c + \overrightarrow{2}$
   (d) $\overleftarrow{5} = \overrightarrow{2} + d$
   (e) $e + e = \overleftarrow{8}$
   (f) $f + \overrightarrow{3} = \overrightarrow{4} + \overrightarrow{3}$
   (g) $\overleftarrow{13} + g = \overleftarrow{13} + \overrightarrow{7}$
   (h) $(h + \overrightarrow{23}) + \overleftarrow{23} = \overrightarrow{8}$
   (i) $\overrightarrow{17} + (i + \overleftarrow{17}) = \overrightarrow{23}$
   (j) $\overrightarrow{23} + \overrightarrow{2.3} = j$
   (k) $\overrightarrow{23} + k = \overrightarrow{2.3}$

4.5 The Magnitude of a Directed Number

The directed numbers encountered by Fred, Jim and Jack while searching for treasure, turned out to be translations in various directions. For example, the directed numbers $\overrightarrow{3}$, $\overleftarrow{3}$, $3\uparrow$, and $3\downarrow$ correspond respectively to translations toward the east, west, north and south. These are certainly different translations, yet they all have something in common--they map any given point onto a point which is 3 units away.

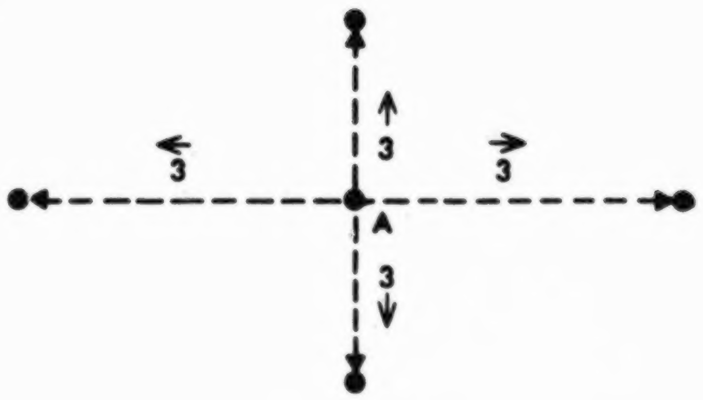


Figure 4.8

We shall express this fact by saying that 3 is the magnitude of each of the four directed numbers $\overrightarrow{3}, \overleftarrow{3}, 3\uparrow, 3\downarrow$.

It is customary to denote the magnitude of a directed number by using vertical bars as follows:

$$|\overrightarrow{3}| = 3 \qquad |3\uparrow| = 3 \qquad |\overrightarrow{0}| = |\overleftarrow{0}| = |0\downarrow| = |0\uparrow| = 0$$
$$|\overleftarrow{3}| = 3 \qquad |3\downarrow| = 3$$

In general if b is any directed number, the magnitude of b is denoted by |b|. We usually try to express the magnitude of a directed number in terms of the usual familiar numerals. This may require a computation.

For example, let us compute $|\overrightarrow{9} + \overleftarrow{6}|$ (i.e. the magnitude of the sum of 9 and 6). In order to do this, we first compute

$$\overrightarrow{9} + \overleftarrow{6} = \overrightarrow{3}$$
$$\text{Therefore } |\overrightarrow{9} + \overleftarrow{6}| = |\overrightarrow{3}|$$
$$\text{i.e. } |\overrightarrow{9} + \overleftarrow{6}| = 3$$

This expresses the magnitude of $\overrightarrow{9} + \overleftarrow{6}$ as a familiar numeral, namely, "3". We have computed $|\overrightarrow{9} + \overleftarrow{6}|$ to be 3.

As a second illustration let us compute $|\overrightarrow{9}| + |\overleftarrow{6}|$ (this means the sum of the magnitudes of 9 and 6). Here we first observe that

$$|\overrightarrow{9}| = 9$$
$$\text{and } |\overleftarrow{6}| = 6$$
$$\text{Hence } |\overrightarrow{9}| + |\overleftarrow{6}| = 9 + 6$$
$$\text{or } |\overrightarrow{9}| + |\overleftarrow{6}| = 15$$

Notice that $|\overrightarrow{9} + \overleftarrow{6}|$ is not the same as $|\overrightarrow{9}| + |\overleftarrow{6}|$. This shows that if b and c are directed numbers then |b + c| need not be the same as |b| + |c|. (Under what circumstances will they be the same?)

4.6 Exercises

I. Compute each of the following:

1. $\lvert\overrightarrow{7}\rvert$	15. $\lvert\overrightarrow{7} + \overrightarrow{3}\rvert$
2. $\lvert\overrightarrow{5} + \overrightarrow{2}\rvert$	16. $\lvert\overrightarrow{7} - \overrightarrow{3}\rvert$
3. $\lvert\overrightarrow{5} + \overleftarrow{2}\rvert$	17. $\lvert\overrightarrow{7} - \overleftarrow{3}\rvert$
4. $\lvert\overleftarrow{5} + \overrightarrow{2}\rvert$	18. $\lvert\overrightarrow{7} + \overleftarrow{3}\rvert$
5. $\lvert\overleftarrow{5} + \overleftarrow{2}\rvert$	19. $\lvert\overrightarrow{7}\rvert + \lvert\overleftarrow{3}\rvert$
6. $\lvert\overrightarrow{5} - \overrightarrow{2}\rvert$	20. $\lvert\overrightarrow{7} + \overrightarrow{7}\rvert$
7. $\lvert\overrightarrow{5} - \overleftarrow{2}\rvert$	21. $\lvert\overrightarrow{7} + \overleftarrow{7}\rvert$
8. $\lvert\overleftarrow{5} - \overrightarrow{2}\rvert$	22. $\lvert\overrightarrow{7} - \overrightarrow{7}\rvert$
9. $\lvert\overleftarrow{5} - \overleftarrow{2}\rvert$	23. $\lvert\overrightarrow{7} - \overleftarrow{7}\rvert$
10. $\lvert\overrightarrow{5}\rvert + \lvert\overrightarrow{2}\rvert$	24. $\lvert\overrightarrow{7}\rvert + \lvert\overleftarrow{7}\rvert$
11. $\lvert\overrightarrow{5}\rvert + \lvert\overleftarrow{2}\rvert$	25. $\lvert\overleftarrow{7}\rvert + \lvert\overrightarrow{7}\rvert$
12. $\lvert\overrightarrow{5}\rvert - \lvert\overrightarrow{2}\rvert$	26. $\lvert\overrightarrow{7.5} - \overrightarrow{22.5}\rvert$
13. $\lvert\overrightarrow{5}\rvert - \lvert\overleftarrow{2}\rvert$	27. $\lvert\overrightarrow{22.5} - \overrightarrow{7.5}\rvert$
14. $\lvert 5\uparrow\rvert + \lvert\overrightarrow{2}\rvert$	28. $\lvert\overrightarrow{22.5} + \overrightarrow{7.5}\rvert$

II. 1. What meaning would you give to $|\overrightarrow{3} + 4\uparrow|$? (It is possible to give various meanings)

Using whatever meaning you adopted in 1, Compute:

2. $|\overrightarrow{3} + 4\downarrow|$
3. $|4\uparrow + \overrightarrow{3}|$
4. $|\overrightarrow{3} - 4\uparrow|$
5. $|4\downarrow - \overrightarrow{3}|$

4.7 A Flow Chart for Addition

We have interpreted addition of directed numbers as a composite mapping formed by combining translations, one following another. In the rest of this chapter we shall keep things simple by confining our directed numbers to translations in either of two opposite directions, for example, east and west only. There is much that we can learn from this simpler situation. It will in fact enable us to obtain a deeper understanding of numbers.

Although we shall continue to speak about directed numbers, bear in mind that we are going to confine our attention, until further notice, solely to these "linearly" directed numbers. If you wish, you may think of them as "east-west" translations or perhaps as "left-right" translations. Let us call this set of linearly directed numbers D. Addition of directed numbers in D is fairly easy. Let us look at a few simple examples:

There are just two important cases

A: Addends having the same direction

$$\overrightarrow{5} + \overrightarrow{2} = \overrightarrow{7} \qquad \overleftarrow{5} + \overleftarrow{2} = \overleftarrow{7}$$

B: Addends having opposite directions

$$\overrightarrow{5} + \overleftarrow{2} = \overrightarrow{3} \qquad \overleftarrow{5} + \overrightarrow{2} = \overleftarrow{3}$$

These illustrations suggest the following generalization:

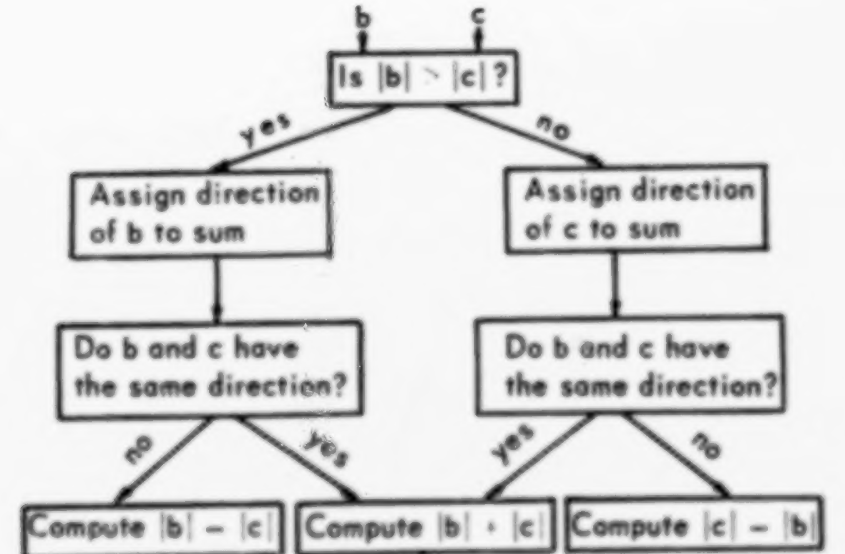
If b and c are any directed numbers in D then:

A. The sum b + c has the same direction as the addend of larger magnitude (If the addends have the same magnitude, then the direction of either addend may be used)

B. The magnitude of b + c is:
  1. The sum of their magnitudes whenever b and c have the same direction.
  2. The difference of their magnitudes (the larger-the smaller) whenever b and c have opposite directions.

This rule for computing the sum b + c of directed numbers b and c is also neatly formulated in the form of a flow chart.

COMPUTING THE SUM OF A PAIR OF DIRECTED NUMBERS IN D


b
c
Is |b| > |c| ?
yes
no
Assign direction of b to sum
Assign direction of c to sum
Do b and c have the same direction?
Do b and c have the same direction?
no
yes
yes
no
Compute |b| − |c|
Compute |b| + |c|
Compute |c| − |b|
Assign this value as the magnitude of b + c

Flow charts like these are of great importance in preparing a program of instructions to be executed by a computer in carrying out a computation. Let us see how this flow chart can serve as guide for a specific computation. Suppose we want to compute

$$\overleftarrow{6} + \overrightarrow{9}$$

In this problem $b = \overleftarrow{6}$ and $c = \overrightarrow{9}$.

(1) Is |b| > |c| ? In this case $|\overleftarrow{6}| < |\overrightarrow{9}|$ (because 6 < 9). Hence the answer is NO.

(2) Therefore, following the branch marked NO, we assign the direction of $\overrightarrow{9}$ to the sum $\overleftarrow{6} + \overrightarrow{9}$.

(3) Do b and c have the same direction? In this case $\overleftarrow{6}$ and $\overrightarrow{9}$ have opposite directions. Hence the answer is again NO.

(4) We therefore continue along the branch marked NO and compute $|c| - |b|$ or $|\overrightarrow{9}| - |\overleftarrow{6}| = 9 - 6 = 3$

(5) Assign 3 as the magnitude of b + c that is $|\overleftarrow{6} + \overrightarrow{9}| = 3$. Combining the results obtained in steps (2) and (5) we get $\overleftarrow{6} + \overrightarrow{9} = \overrightarrow{3}$.

Other possible cases, involving other paths through the flow chart, are indicated in the following exercises.

4.8 Exercises

1. Using the flow chart as a guide, compute each of the following:
   - a. $\overrightarrow{7} + \overrightarrow{2}$
   - b. $\overrightarrow{7} + \overleftarrow{2}$
   - c. $\overrightarrow{7} + \overleftarrow{7}$
   - d. $\overleftarrow{7} + \overrightarrow{7}$
   - e. $\overleftarrow{7} + \overrightarrow{5}$
   - f. $\overrightarrow{5} + \overleftarrow{7}$

2. By use of the flow chart try to give an argument to prove that whenever b and c are linearly directed numbers

$$b + c = c + b$$

Assume that addition of ordinary numbers is commutative.

3. We have defined addition of directed numbers as a composition of translations. Explain how the flow chart above could be used to define addition of linearly directed numbers.

4. If b and c are linearly directed numbers, construct a flow chart for computing b − c.

4.9 Subtraction of Directed Numbers

We have already considered the problem of subtracting directed numbers in Exercise 9 of Section 4.1.

For example to compute:

$$\overrightarrow{5} - \overrightarrow{2} = \square \quad ?$$

we think of the corresponding addition sentence

$$\overrightarrow{5} = \square + \overrightarrow{2}$$

and immediately recognize that the missing addend must be $\overrightarrow{3}$. Hence

$$\overrightarrow{5} - \overrightarrow{2} = \overrightarrow{3}$$

The answer to this problem is particularly simple because the problem involves only "linearly" directed numbers (all of which are translations in the same direction). The problem is in fact quite similar to the corresponding subtraction problem involving whole numbers, namely

$$5 - 2 = 3$$

However, suppose we change our whole number problem to read:

$$2 - 5 = \square \quad ?$$

The corresponding addition sentence

$$2 = \square + 5$$

is not very helpful because there is no whole number which can serve as a missing addend. In fact the subtraction problem "2 − 5 = □ " has no solution in the realm of whole numbers. "2 − 5" does not name a whole number. But how about the corresponding subtraction problem for linearly directed numbers, namely:

$$\overrightarrow{2} - \overrightarrow{5} = \square$$

If we look at the equivalent addition sentence

$$\overrightarrow{2} = \square + \overrightarrow{5}$$

we see immediately that there is a perfectly good solution, namely the directed number $\overleftarrow{3}$. We express this by writing

$$\overrightarrow{2} - \overrightarrow{5} = \overleftarrow{3}$$

In other words, "$\overrightarrow{5} - \overrightarrow{2}$" and "$\overrightarrow{2} - \overrightarrow{5}$" both name directed numbers.

We can describe this situation in another way. If b and c are whole numbers, the equation

$$x + b = c$$

does not always have a solution. More specifically, whenever b is larger than c, the equation x + b = c does not have a solution in the realm of whole numbers. (For instance, in our example above, the equation

$$x + 5 = 2$$

did not have a whole number solution). However, suppose that b and c are linearly directed numbers. Then the equation

$$x + b = c$$

always has a solution which is a directed number.

Subtraction is therefore a binary operation in the realm of linearly directed numbers but not in the realm of whole numbers.

4.10 Exercises

1. (a) Compute each of the following differences. After computing all of them, try to formulate a generalization. Try to give an argument as to why your generalization holds:

(1) 6 − 3	$\overrightarrow{6} - \overrightarrow{3}$
(2) 7 − 4	$\overrightarrow{7} - \overrightarrow{4}$
(3) 8 − 5	$\overrightarrow{8} - \overrightarrow{5}$
(4) 18 − 15	$\overrightarrow{18} - \overrightarrow{15}$

(b) Compute each of the following differences and show how your generalization (in (a)) might have been used to simplify the computation.

(1) 326 − 97
(2) 6 3/8 − 2 7/8
(3) 7' 5'' − 2' 10'
(4) 9 lbs 3 oz. − 5 lbs, 14 oz.

2. (a) Compute each of the following differences. After computing all of them, try to formulate a generalization. Try to give an argument as to why your generalization holds.

(1) 5 − 2	$\overrightarrow{5} - \overrightarrow{2}$
(2) 10 − 4	$\overrightarrow{10} - \overrightarrow{4}$
(3) 15 − 6	$\overrightarrow{15} - \overrightarrow{6}$
(4) 150 − 60	$\overrightarrow{150} - \overrightarrow{60}$

(b) Compute each of the following differences and show how your generalization (in a) might have been used to simplify the computation.

(1) (47 × 28) − (46 × 28)
(2) (47 × 28) − (36 × 28)
(3) (7 1/2 × 28) − (6 1/2 × 28)
(4) (7 1/2 × 28) − (5 1/2 × 28)
(5) 7 1/2 × 28 1/2) − (5 1/2 × 28 1/2)

3. (a) Solve each of the following equations for n. After solving all of them try to formulate a generalization and then try to give an argument to support your generalization.

(1) n + 2 = 8 + 2	n − 2 = 8 − 2
(2) n + d = 8 + d	n − d = 8 − d
(3) 9 + 3 = n + 3	9 − 3 = n − 3
(4) 9 + d = n + d	9 − d = n − d

(b) Replace each of the following equations by an equivalent equation, in which you can use the generalization you formulated in (a).

(1) n + 3 = 11	n − 3 = 5
(2) n + 13 = 33	n − 13 = 7
(3) 9 = n + 4	1 = n − 4
(4) 27 = n + 9	9 = n − 9

4.11 An Isomorphism Between W and $\overrightarrow{W}$

Although subtraction is not always possible in the realm of whole numbers, we have seen that subtraction is always possible in the set of linearly directed numbers D. Since we can do things with the directed numbers of D which were not possible with ordinary whole numbers, we are inclined to regard this system of directed numbers namely (D, +) as somehow more satisfactory than the system of whole numbers (W, +).

In support of this attitude we shall now cite another important feature of the system of linearly directed numbers (D, +). Temporarily, let us restrict our attention to those linearly directed numbers of D which have just one of the two possible directions of D. (we might choose, for example, only translations toward the right) Let us also require that the magnitudes of our directed numbers be whole numbers. With these restrictions we obtain the following subset of D:

$$\{\overrightarrow{0}, \overrightarrow{1}, \overrightarrow{2}, \overrightarrow{3}, \overrightarrow{4}, \dots\}$$

Let us call this subset $\overrightarrow{W}$ (Read: "W to the right"). Its members are translations all of which have the same direction (to the right) and a whole number magnitude. The set of directed numbers $\overrightarrow{W}$ behaves, as far as addition and subtraction are concerned, very much like the set of ordinary whole numbers W:

$$\{0, 1, 2, 3, 4, \dots\}.$$

An addition of two directed numbers of $\overrightarrow{W}$, for example:

$$\overrightarrow{4} + \overrightarrow{3} = \overrightarrow{7}$$

looks very much like the addition of the corresponding whole numbers of W:

$$4 + 3 = 7$$

Similarly, a subtraction involving directed numbers, such as:

$$\overrightarrow{4} - \overrightarrow{3} = \overrightarrow{1}$$

looks very much like the corresponding subtraction problem in W:

$$4 - 3 = 1$$

Now, certain subtractions involving whole numbers such as:

$$3 - 4$$

are not possible, because "3 − 4" does not name a whole number. The corresponding subtraction problem for directed numbers

$$\overrightarrow{3} - \overrightarrow{4}$$

has the solution $\overleftarrow{1}$, but notice that this solution ($\overleftarrow{1}$) is not a member of $\overrightarrow{W}$. The subtraction is possible only if we go "outside" of $\overrightarrow{W}$ (to a larger set such as D), and are willing to accept a translation in a new direction (toward the left) as the answer to our subtraction problem. Strictly speaking the subtraction $\overrightarrow{3} - \overrightarrow{4}$ is no more possible in the directed number system ($\overrightarrow{W}$, +), than the subtraction 3 − 4 in the whole number system (W, +).

If we adopt this point of view, we see that the two number systems (W, +) and ($\overrightarrow{W}$, +) are like identical twins. We can, in fact, set up a the following (one-to-one) mapping from the whole numbers of W, to the directed numbers of W:

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & n & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ \overrightarrow{0} & \overrightarrow{1} & \overrightarrow{2} & \overrightarrow{3} & \dots & \overrightarrow{n} & \dots \end{array}$$

The rule for this mapping is very simple: the image of each whole number n is the directed number $\overrightarrow{n}$.

We now observe that the mapping we have just set up "preserves" both addition and subtraction. This means that if a, b, and c are any whole numbers whatsoever, and if a, b, and c are the corresponding directed numbers under our mapping, then

(1) a + b = c if and only if a + b = c

(2) a − b = c if and only if a − b = c.

Additions and subtractions in either of these number systems are mirrored exactly by corresponding additions and subtractions in the other number system. Mathematicians express this idea by describing the two systems as isomorphic (literally this means "having the same form"). Isomorphisms play a key role in developing a deeper understanding of number and other mathematical ideas.

4.12 Exercises

1. (a) Let us consider the following set of commands:

L= left face
a = about face
r = right face
s = stay (or do nothing)

Let the operation on this set be "followed by" which will be denoted by *. Thus, l*r will mean to execute the command "left face" and then the command "right face." The net result is to arrive at one's original position, which could have been achieved by "doing nothing." This is expressed by writing l*r = s. Complete the table below:

*	s	L	a	r
s	s	L	a	r
L				s
a				
r				

(b) Complete the following addition table for $(Z_4, +)$

+	0	1	2	3
0				
1				
2				
3				

(c) To show that the systems described in (a) and (b) are isomorphic we must find a one-to-one mapping between {s, l, a, r} and {0, 1, 2, 3} that preserves their operations. By this we mean that if we denote the images under this mapping of s, l, a, r by s', l', a', r' (which are, in some order, 0, 1, 2, 3) then for any commands x and y

$$(x * y)' = x' + y'$$

Find such a mapping between {s, l, a, r} and {0, 1, 2, 3}.

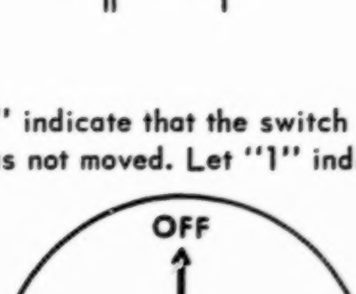
Thus: (l*r)' = (s)' = 0 and l' + r' = 1 + 3 = 0, so that (l*r)' = l' + r'.
Test your isomorphism on l*a, s*r, a*r.

(d) Show that the following set {0, 2, 4, 6} under addition in $(Z_8, +)$ defines another system isomorphic to each of the above systems.

2. (a) Let E stand for any even number and O, any odd number. The following "addition" table show what happens when even and odd numbers are added.
Complete the table:

+	E	O
E	E	
O	O	

(b) Let "0" indicate that the switch shown here was not moved. Let "1" indicate



that the switch was turned to "on" if it was off and to "off" if it was on. Let the operation * be "followed by." Fill in the table:

*	0	1
0		
1		

(c) Show that the systems in (a) and (b) are isomorphic.

(d) Show that the system defined by the following "multiplication" table is isomorphic to the system described in (a).

x	→	←
→	→	←
←	←	→

3. Try to describe in your own words when two systems $(S, *_1)$ and $(T, *_2)$ are isomorphic.

4.13 The Set of Integers Z

The fact that the whole numbers W and the "one-way" directed numbers $\overrightarrow{W}$ are isomorphic with respect to both addition and subtraction suggests a way to remedy the deficiency of the whole number system in regard to subtraction.

You will recall that we obtained $\overrightarrow{W}$ as a subset of a larger set of directed numbers, namely our set D of linearly directed numbers, by imposing two conditions; we required that:

A. All the directed numbers of $\overrightarrow{W}$ have the same direction.

B. All the directed numbers of $\overrightarrow{W}$ have whole number magnitudes.

Suppose we relax the first requirement and impose only the second restriction. This means we select from D all those directed numbers that have whole number magnitudes regardless of their direction. We obtain a larger subset consisting of the following set of directed numbers

$$\overrightarrow{0}, \overrightarrow{1}, \overrightarrow{2}, \overrightarrow{3}, \dots, \overrightarrow{n}, \dots$$
$$\overleftarrow{0}, \overleftarrow{1}, \overleftarrow{2}, \overleftarrow{3}, \dots, \overleftarrow{n}, \dots$$

We designate this new set by $\overleftrightarrow{W}$ (read "W both ways"). Notice that $\overleftrightarrow{W}$ contains $\overrightarrow{W}$ as a subset. Each directed number of $\overleftrightarrow{W}$ has a whole number magnitude, but $\overleftrightarrow{W}$ now includes translations toward the left as well as translations toward the right. In fact, for each whole number n, the set $\overleftrightarrow{W}$ contains both the directed number $\overrightarrow{n}$ and its "opposite" $\overleftarrow{n}$. These opposite translations form an interesting subset, namely

$$\{\overleftarrow{0}, \overleftarrow{1}, \overleftarrow{2}, \overleftarrow{3}, \dots\}$$

which we can conveniently designate by $\overleftarrow{W}$ (read "W left"). Observe that $\overrightarrow{W}$ and $\overleftarrow{W}$ have exactly one member in common, namely the zero translation (which is the same, left or right):

$$\overrightarrow{0} = \overleftarrow{0}$$

We can think of the set $\overleftrightarrow{W}$ as an extension or enlargement of the set $\overleftrightarrow{W}$ formed by uniting the set $\overrightarrow{W}$ with $\overleftarrow{W}$. It is the enlarged set $\overleftrightarrow{W}$ which enables us to overcome the difficulty we encountered concerning subtraction of whole numbers. $\overleftrightarrow{W}$ contains the set $\overrightarrow{W}$ which behaves like an identical twin to W, because of the isomorphism of $\overrightarrow{W}$ and W as far as addition and subtraction are concerned. A subtraction of whole numbers such as 3 − 4, which is impossible in the system (W, +) corresponds (because of the isomorphism) to the subtraction of directed numbers $\overrightarrow{3} - \overrightarrow{4}$ which is also impossible in the system ($\overrightarrow{W}$, +). However the subtraction $\overrightarrow{3} - \overrightarrow{4}$ is no longer impossible in the larger system ($\overleftrightarrow{W}$, +). In fact, if x and y are any directed numbers in $\overleftrightarrow{W}$, the subtraction x − y is always possible in ($\overleftrightarrow{W}$, +). If, in particular, $\overrightarrow{p}$ and $\overrightarrow{q}$ are directed numbers in W, $\overrightarrow{p} - \overrightarrow{q}$ can therefore always be obtained by "escaping" if necessary, from $\overrightarrow{W}$ to $\overleftrightarrow{W}$. This sort of stunt is not possible with whole numbers. If p and q are whole numbers and if "p − q" does not name a whole number, we cannot obtain a value for "p − q" by escaping from W, because there is no place to escape. However the "twin brother" of W, namely $\overrightarrow{W}$, happens to be "embedded" in a very lovely larger set $\overleftrightarrow{W}$, where subtraction is always possible. Any addition or subtraction in W is mirrored by a corresponding addition or subtraction in $\overrightarrow{W}$ by virtue of the isomorphism which connects the two number systems. So we might just as well perform these calculations in $\overrightarrow{W}$. However, it is even better still to perform them in $\overleftrightarrow{W}$, because we can then obtain an answer to any subtraction problem (as well as any addition problem.

By constructing the set $\overleftrightarrow{W}$ we have in a sense extended the whole number system (W, +) to a new system which is more satisfactory than (W, +). We might say, exuberantly "Anything W can do, $\overleftrightarrow{W}$ can do better" provided we do not take this too literally.

The set $\overleftrightarrow{W}$, together with the operations of addition and subtraction of the directed numbers in $\overleftrightarrow{W}$, makes a very nice number system. But we should not forget that the "numbers" in $\overleftrightarrow{W}$ have a strong geometrical flavor - we visualize them as translations along certain directions. Many mathematicians prefer to think about numbers from a more abstract or algebraic point of view, apart from any visual or physical interpretation, helpful as these interpretations might be. It would be nice if we could obtain our extended number system by working only with the whole numbers and their properties. In seeking to extend the whole number system as we have done here, we really should not stop with the system of directed numbers ($\overleftrightarrow{W}$, +). We should go on to build a "pure" number system directly out of the whole number system. We shall indicate how this is done later on (see Section 4.15). The pure number system so obtained will be an isomorphic twin of the system ($\overleftrightarrow{W}$, +) in the same way that the whole number system is isomorphic to the "one-way" directed number system (W, +).

The numbers in this new "pure" number system are called integers. The set of all integers is denoted by Z, and addition of integers, which will mirror the addition of directed numbers, will be denoted by the usual + sign.

Because of the isomorphism of the new system (Z, +) and the system (W, +), the members of Z, that is the integers themselves, are designated by symbols which use familiar numbers in such a way as to derive the greatest benefit from the isomorphism. The integers are accordingly represented as follows:

$$Z = \left\{ \begin{matrix} {}^{+}0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots, {}^{+}n, \ldots \\ {}^{-}0, {}^{-}1, {}^{-}2, {}^{-}3, \ldots, {}^{-}n, \ldots \end{matrix} \right\}$$

where we read "$^{+}2$" as "positive two," "$^{-}3$" as "negative three," etc., and it is agreed that the integer $^{+}0$ is the same as the integer $^{-}0$.

Although we shall see later on, how to define the integers directly in terms of the whole numbers, at present it will suffice to think of the integers as essentially the same as the linearly directed numbers in W, but with slightly different names. Actually we are thinking of a correspondence (one-to-one mapping) between the directed numbers in W and the integers in Z, illustrated by

$$\overrightarrow{3} \longleftrightarrow {}^{+}3, \quad \overleftarrow{3} \longleftrightarrow {}^{-}3$$
$$\overrightarrow{7} \longleftrightarrow {}^{+}7, \quad \overleftarrow{7} \longleftrightarrow {}^{-}7$$
$$\overrightarrow{0} \longleftrightarrow {}^{+}0, \quad \overleftarrow{0} \longleftrightarrow {}^{-}0$$

---- etc. ----

The general rule for this mapping can be expressed as follows for every whole number n:

$$\overrightarrow{n} \longleftrightarrow {}^{+}n \text{ and } \overleftarrow{n} \longleftrightarrow {}^{-}n$$

Addition and subtraction of integers in Z will simply mirror the addition and subtraction of the corresponding directed numbers in W. Here are a few examples:

Example 1. Compute $(^{+}7) + (^{-}3)$

Solution

$(^{+}7) + (^{-}3)$ corresponds to $\overrightarrow{7} + \overleftarrow{3}$

But $\overrightarrow{7} + \overleftarrow{3} = \overrightarrow{4}$, which corresponds to $^{+}4$

Hence $(^{+}7) + (^{-}3) = {}^{+}4$

Example 2. Compute $(^{-}8) + (^{+}5)$

Solution

$(^{-}8) + (^{+}5)$ corresponds to $\overleftarrow{8} + \overrightarrow{5}$

But $\overleftarrow{8} + \overrightarrow{5} = \overleftarrow{3}$, which corresponds to $^{-}3$.

Hence $(^{-}8) + (^{+}5) = {}^{-}3$

Example 3. Compute $(^{-}6) - (^{+}2)$

Solution

$(^{-}6) - (^{+}2)$ corresponds to $6 - 2$

But $6 - 2 = 8$ (because $6 = 8 + 2$), which corresponds to $^{-}8$. Hence

$(^{-}6) - (^{+}2) = {}^{-}8$

Before going further, you should practice computing with integers until you are skillful. This will be no problem if you are good at computing with directed numbers.

4.14 Exercises

1. Under the isomorphism between (Z, +) and (W, +) state which element corresponds to each of the following

$^{+}8, {}^{-}8, 0, {}^{+}100, {}^{-}100, 23, 23, 0, 0$

2. Under the isomorphism between (Z, +) and (W, +) write the addition or subtraction expression that corresponds to each of the following and compute the value of each:

(a) $7 + 2$
(b) $7 + 2$
(c) $^{+}7 + {}^{+}3$
(d) $^{-}7 + {}^{-}3$
(e) $^{-}9 + {}^{+}4$
(f) $^{-}9 + {}^{-}4$
(g) $^{+}23 + {}^{-}7$
(h) $^{-}23 + {}^{+}7$
(i) $7 - 2$
(j) $7 - 2$
(k) $^{+}7 - {}^{+}3$
(l) $^{-}7 - {}^{-}3$
(m) $^{-}9 - {}^{+}4$
(n) $^{-}9 - {}^{-}4$
(o) $^{+}23 - {}^{-}7$
(p) $^{-}23 - {}^{+}7$

3. Work all of the following exercises and then try to formulate a generalization suggested by them.

(a) Express each difference as a sum:

(1) $7 - 2$	$^{+}7 - {}^{+}2$
(2) $7 - 2$	$^{+}7 - {}^{-}2$
(3) $2 - 7$	$^{+}2 - {}^{+}7$
(4) $2 - 7$	$^{-}2 - {}^{+}7$
(5) $2 - 7$	$^{-}2 - {}^{-}7$
(6) $b - c$	$^{+}b - {}^{+}c$
(7) $b - c$	$^{+}b - {}^{-}c$
(8) $b - c$	$^{-}b - {}^{+}c$

(9) $b - c$ $\quad -b - {}^-c$

(b) Compute each of the differences in (a) Exercises (1) – (5).

4. Answer True or False; explain your choice.

(a) Addition is an operation on W.

(b) Addition is an operation on W.

(c) Subtraction is an operation on W.

(d) Subtraction is an operation on W.

(e) Subtraction is an operation on W.

(f) Subtraction is an operation on W.

(g) Subtraction is an operation on Z.

(h) Addition is an operation on Z.

5. Solve:

(a) $n + {}^+2 = {}^+8$

(b) $n + -2 = {}^+8$

(c) $n + {}^+2 = -8$

(d) $n + {}^-2 = -8$

(e) $n + {}^+8 = {}^+2$

(f) $n + -8 = {}^+2$

(g) $n + {}^+8 = -2$

(h) $n + -8 = -2$

6. Solve:

(a) $n - {}^+2 = {}^+8$

(b) $n - -2 = {}^+8$

(c) $n - {}^+2 = -8$

(d) $n - -2 = -8$

(e) $n - {}^+8 = {}^+2$

(f) $n - -8 = {}^+2$

(g) $n - {}^+8 = -2$

(h) $n - -8 = -2$

7. Solve:

(a) ${}^+3 = a + -5$

(b) ${}^+3 = b - -5$

(c) ${}^+3 = -5 - c$

(d) $-3 = d + -5$

(e) $-3 = e - -5$

(f) $-3 = -5 - f$

(g) $-46 = {}^+17 - g$

(h) ${}^+17 = -46 - h$

8. (a) Solve each of the following equations for n. After solving all of them try to formulate a generalization and then try to give an argument to support your generalization.

(1) $n + {}^+2 = -8 + {}^+2$ $\quad n - {}^+2 = -8 - {}^+2$

(2) $n + d = {}^-8 + d$ $\quad n - d = -8 - d$

(3) ${}^+9 + -3 = n + -3$ $\quad {}^+9 - -3 = n - -3$

(4) ${}^+9 + d = n + d$ $\quad {}^+9 - d = n - d$

(5) $n + a = d + a$ $\quad n - a = d - a$

9. Replace each of the following equations by an equivalent equation in which you can use the generalization you formulated in a.

(1) $n + {}^+3 = {}^+11$ $\quad n - {}^+3 = {}^+5$

(2) $n + {}^+3 = -11$ $\quad n - {}^+3 = -17$

(3) $-9 = n + {}^+4$ $\quad -17 = n - {}^+4$

(4) ${}^+27 = n + {}^+9$ $\quad {}^+9 = n - {}^+9$

(5) $n + -3 = {}^+7$ $\quad n - -3 = {}^+2$

(6) $n + -3 = -7$ $\quad n - -3 = -2$

10. Replace each of the following subtraction expressions by an equivalent addition expression:

(a) ${}^+7 - {}^+3,$ $\quad {}^+7 - -3$

(b) $-7 - {}^+3,$ $\quad -7 - -3$

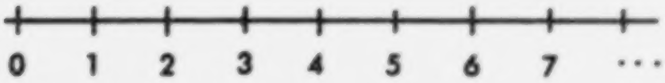
(c) ${}^+0 - {}^+3,$ $\quad {}^+0 - -3$

4.15 Construction of Z from W

Although we have been working with the system of integers, we have not yet really defined what the integers are. We pointed out (in Section 4.13) that it would be nice to build the integers directly out of the whole numbers, using only properties of the whole numbers. We are now going to describe how this can be done.

Now we want our new number system, the integers, to be isomorphic to the "two-way" directed number system (W, +) which we saw was a nice improvement over the whole numbers and which was easy to visualize. Therefore, we shall adopt a somewhat "sneaky" plan, the idea of which is used quite widely by mathematicians. We shall use directed numbers and their properties to guide the process of constructing the system of integers, but the properties of the final system we obtain containing the integers will depend only on the properties of the whole numbers.

Let us begin with the whole numbers W, visualized by points on a number line



Now any directed number in W, let us say 2, can be visualized as a translation which maps each whole number point onto a new point of the number line located 2 units to the right.

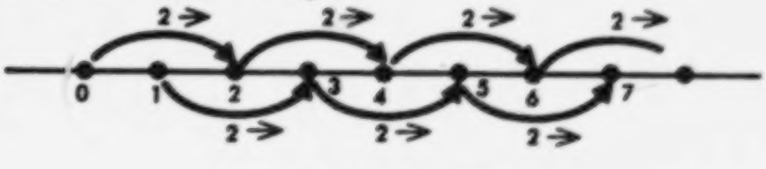


Figure 4.9

The rule for this mapping can be expressed:

$$n \longrightarrow (2+n) \text{ for \underline{all} whole numbers } n.$$

As a result of this mapping each whole number n is paired with another whole number, its image 2 + n, thus creating the following set of ordered pairs of whole numbers:

$$A = \{(0,2), (1,3), (2,4), \ldots, (n, 2+n), \ldots\}$$

The ordered pairs in set A all share the common property that the second number is 2 greater than the first number.

Next, let us select from W the "oppositely" directed number 2. This represents a translation which maps each point of the line onto a new point located 2 units to the left.

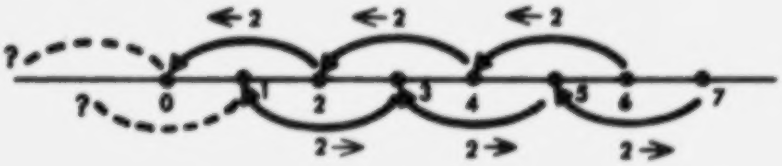


Figure 4.10

Notice, however, that under this mapping only the whole numbers greater than 1 are mapped onto whole numbers. The numbers 0 and 1 do not have a whole number image. As far as whole numbers are concerned, the rule for this mapping could be expressed as follows:

$$(2+n) \longrightarrow n \text{ for all whole numbers } n$$

As a result of this mapping, each whole number 2 + n starting with 2, is paired with another whole number, its image n thus creating the following set of ordered pairs of natural numbers:

$$B = \{(2,0), (3,1), (4,2), \ldots, (2+n,n), \ldots\}$$

The ordered pairs in set B all share the common property that the first number is two larger than the second number.

If we compare set B with set A, we see that these two sets represent inverse mappings. This, of course, is not surprising because the directed numbers 2 and 2 represent inverse translations.

Just as we used 2 and 2 to construct the "inverse" sets A and B, we could use other pairs of "opposite" directed numbers to construct other "inverse" sets. For example, using 3 and 3 we obtain the inverse sets.

$$C = \{(0,3), (1,4), (2,5), \ldots, (n,3+n), \ldots\}$$

$$D = \{(3,0), (4,1), (5,2), \ldots, (3+n,n), \ldots\}$$

where the set C corresponds to the translation (directed number) 3 and the set D corresponds to the translation 3.

In general, corresponding to any whole number d, we can find two "opposite" directed numbers d and d in the set W. We then use the directed number d to construct a set P of ordered pairs of whole numbers

$$P = \{(0,d), (1,d+1), (2,d+2), \ldots, (n,d+n), \ldots\}$$

and we use the directed number d to construct the set Q of inverse pairs:

$$Q = \{(d,0), (d+1,1), (d+2,2), \ldots, (d+n,n), \ldots\}$$

The set P represents a translation mapping W into W, while the set Q represents the inverse mapping.

Now notice that all these new sets, A, B, C, etc., and in general the sets P,Q are constructed solely out of the raw material of the whole number system (W, +). The members of P and Q are ordered pairs of whole numbers like (n,d+n) or (d+n,n) where n is a whole number and d + n is the sum of whole numbers d and n. Each of these newly constructed sets corresponds of course to a unique directed number, thus:

$$A \longleftrightarrow \overrightarrow{2} \qquad B \longleftrightarrow \overleftarrow{2}$$

$$C \longleftrightarrow \overrightarrow{3} \qquad D \longleftrightarrow \overleftarrow{3}$$

and for each whole number d:

$$P \longleftrightarrow \overrightarrow{d} \qquad Q \longleftrightarrow \overleftarrow{d}$$

However we do not need to resort to directed numbers to construct these new sets. All that we require for their formation is the availability of the whole numbers, W and the operation +, for addition of whole numbers. We shall therefore seize hold of these new sets and call them integers.

We begin now to see what the integers are, from a purely mathematical point of view. Each integer is really a whole class (set) of ordered pairs of whole numbers. Within each one of these classes, the ordered pairs of whole numbers have a common property, which distinguishes these ordered pairs from the ordered pairs which make up a different integer. What is this common property?

For a class such as P, each ordered pair has the form

$$(n,d+n)$$

which means that the second whole number exceeds the first whole number by d. Each whole number d therefore determines a different class of type P, i.e. a different integer. The integers A and C are of this type, because for every ordered pair in set A, the second whole number is 2 more than the first whole number; for every ordered pair in set C, the second whole number is 3 more than the first whole number. Because of this property we call A and C positive integers and we rename them as follows:

$$A = {}^{+}2,\ C = {}^{+}3$$

In general, for each whole number d, we rename the set P as ${}^{+}d$.

Similarly in a class of type Q, each ordered pair has the form

$$(d+n,n)$$

which means that the first whole number exceeds the second whole number by d. The integers B and D are of this type because for every ordered pair in B, for example, the first whole number exceeds the second whole number by 2; for every ordered pair in D, the first whole number exceeds the second whole number by 3. Because of this property we call B and D negative integers and we re-name them as follows

$$B = -2,\ D = -3$$

In general, for each whole number d we rename the set Q as $-d$.

Thus within our newly constructed set of integers we find positive and negative integers. These two subsets of the integers mirror the two subsets of "one-way" directed numbers W and W which together made up the set of two-way directed numbers W.

To complete our description of the integers as a number system we must also define an addition operation for addition of integers which will correspond to the addition operation for directed numbers. Once again, although we use the directed numbers as a guide in constructing our definition, the final definition will refer only to the integers themselves and the whole numbers from which they are built. See Exercise 5. below.

4.16 Exercises

1. For each of the following sets decide whether its ordered pairs can be those of an integer. If so, give the integer. If not, say "no."

(a) I (0,3), (3,6), (6,9), (9,12)I

(b) I (0,3), (1,4), (2,5), (3,6)I

(c) I (0,3), (3,0), (0,4), (4,0)I

(d) I (3,0), (4,1), (5,2), (6,3)I

(e) I (7,5), (5,3), (2,0), (3,1)I

(f) I (0,0), (1,1), (2,2), (3,3)I

(g) I (6,5), (2,3), (4,3), (1,2)I

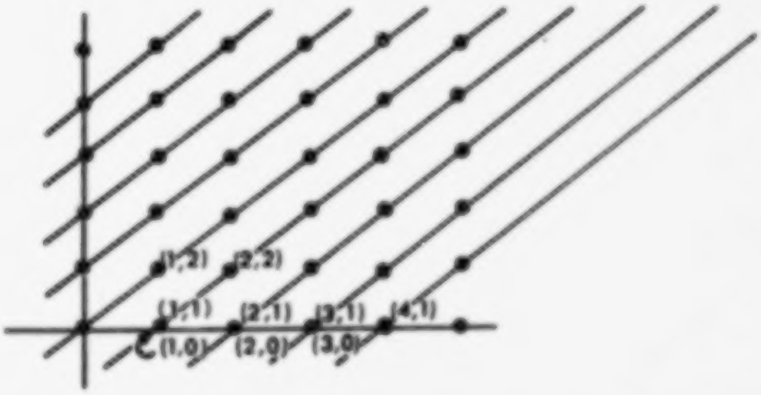
2. Give a set of 4 ordered pairs for each of the following integers:

(a) ${}^{+}7$ (d) ${}^{+}9$

(b) -7 (e) -9

(c) ${}^{+}0$ (f) ${}^{+}16$

3. Some of the points in the figure below have been assigned ordered pair. (a) try to recognize a pattern and use your pattern to assign ordered pairs to the remaining points shown. (b) Try to discover a pattern for all the points on each diagonal line. What integer does each line suggest?



4. In the figure for Ex. 3, draw the other diagonal lines. (For example, the line containing the points for (1,2), (2,1), (3,0).) What pattern do you observe for the ordered pairs on each of these diagonal lines?

5. We are going to define addition of two ordered pairs, but first let us give some examples.

$$(0,3) + (0,4) = (0,7)$$

$$(1,3) + (1,4) = (2,7)$$

$$(7,2) + (1,3) = (8,5)$$

In general, if a,b,c are any whole numbers,

$$(a,b) + (c,d) = (a+c,b+d).$$

a) Using this deinition of addition, select any two ordered pairs one from ${}^{+}3$ and one from ${}^{+}4$. Is their sum in ${}^{+}7$? Try two other ordered pairs. Is their sum in ${}^{+}7$? Try to show that this will always be the case: any ordered pair in ${}^{+}3$ added to any ordered pair on ${}^{+}4$ will give a sum that is an ordered pair in ${}^{+}7$.

b) Do the same for -3, -4, and -7.

c) Do the same for -5, ${}^{+}2$, and -3.

4.17 Ordering the Integers

A very important idea in mathematics is the recognition that numbers can often be compared with regard to "greatness". We say that 7 is greater than 3 and write "$7 > 3$". The same relationship is expressed by saying that 3 is less than 7 and write "$3 < 7$". We would like to be able to compare integers according to "greatness" in a similar way. For example, which is the greater $^{+}7$ or $^{+}3$, $^{+}7$ or -3, -7 or -3? In this section we shall learn how to answer such questions.

Let us look more closely at the idea of "being greater". We think of 7 as being greater than 3 because we must add something to 3 to obtain 7, namely, $7-3$ or 4. Moreover, 3 is not greater than 7 because there is no whole number (not even 0) which when added to 7 gives a sum of 3. In fact, "$3 - 7$" names no whole number. We sould like to regard $^{+}7$ as greater than $^{+}3$, greater than -3, and even greater than -10. How shall we decide when one integer is greater than another integer while retaining the order $^{+}7 > {}^{+}3$, $^{+}7 > -3$, $^{+}7 > -10$? Let us examine the differences.

$$^{+}7 - {}^{+}3 = {}^{+}4$$

$$^{+}7 - -3 = {}^{+}10$$

$$^{+}7 - -10 = {}^{+}17$$

You notice that the differences are all positive integers. We use this idea to decide when integer b is greater than integer c. Let us temporarily call an integer "strictly" positive if it is a positive integer other than $^{+}0$.

> Integer b is greater than integer c
> if b−c is a strictly positive integer.

If integer b is greater than integer c we write "$b > c$". We also express this relationship by saying "c is less than b" and write "$c < b$".

Let us illustrate this idea now by some examples.

Example 1: Which is greater $^{+}2$ or -3? $^{+}2 - -3 = {}^{+}5$ which is a strictly positive integer. Hence $^{+}2 > -3$. Suppose we had tried $-3 - {}^{+}2$, what kind of integer is this difference?

Example 2: Which is greater -2 or -3? $-2 - -3 = {}^{+}1$ which is a strictly positive integer. Hence $-2 > -3$. If we had tried $-3 - -2$ we would have obtained $-3 - -2 = -1$ which is a negative integer.

4.18 Exercises

1. Find the greater integer for each of the following pairs and justify your selection.

(a) -6, $^{+}2$
(b) $^{+}6$, -2
(c) -6, -2
(d) $^{+}0$, -1
(e) -0, $^{+}1$
(f) -6, -7

2. List the following integers in order of greatness starting with the smallest integer at the left.

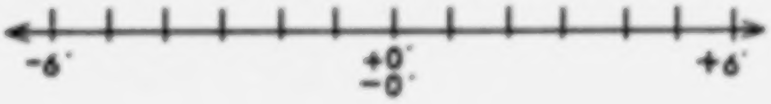
$^{+}2$, -2, $^{+}3$, -5, $^{+}0$, -1, $^{+}4$, -4, -3, $^{+}5$, $^{+}1$

3. If a−b is a "strictly" negative integer, which is the greater integer, a or b?

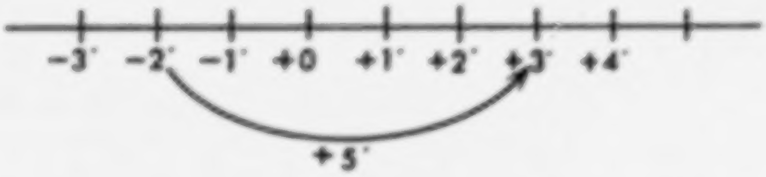
4. Insert the correct symbol =, >, or < whichever fits.

(a) $^{+}7 + -3$ $^{+}7$
(b) $^{+}7 - {}^{+}3$ $^{+}7$
(c) $^{+}7 - {}^{+}3$ $^{+}4$
(d) $-7 + {}^{+}3$ -7
(e) $-7 + {}^{+}3$ -4
(f) $-7 - -3$ $^{+}4$
(g) $-7 + -3$ -7
(i) $^{+}7 - {}^{+}8$ $^{+}1$
(j) $-7 + -9$ -7
(k) $-9 - -7$ $^{+}1$
(l) $^{+}23 - -9$ $^{+}25$
(m) $-23 + {}^{+}9$ -14
(n) $-23 - {}^{+}9$ -14
(o) $-23 - -9$ -14

5. Assign an integer to each point marked on this line which has not already been assigned an integer, in order of greatness.



6. A subtraction computation can be pictured on the number line in the following way. For example, consider the difference $^{+}3 - -2$. Locate the points for $^{+}3$ and -2. Note the number of units between these two points, 5. A direction to the right will be called positive and given the sign "+" while a direction to the left will be called negative and be given the sign "−". The direction from the point for -2 to the point for $^{+}3$ is toward the right, so the difference is positive ($^{+}$), and hence $^{+}3 - -2 = {}^{+}5$.



For each of the following differences, draw such a diagram and indicate the computed value.

(a) $-3 - {}^{+}2$
(b) $-3 - -1$
(c) $^{+}2 - -3$
(d) $-1 - -3$
(e) $-4 - -1$
(f) $^{+}4 - -4$
(g) $^{+}4 - {}^{+}5$
(h) $^{+}4 - -2$

7. By looking at a number line, how can you tell at a glance that $-3 < {}^{+}2$?

8. Try to find a way in which the number line can be used to compute the sum of two integers.

4.19 The Absolute Value of Integers

We have already seen how the notion of magnitude of a directed number was helpful in setting up a flow chart for computing a sum. There is a completely analogous notion available for integers, except that we generally use the words "absolute value" rather than "magnitude". To each integer we shall find a corresponding whole number, illustrated by the following examples:

The absolute value of $^{+}3$ is 3.
The absolute value of $^{-}3$ is 3.

We shall use the very same symbol, the vertical bars, that we used for our directed numbers. The above examples may now be written

$|^{+}3| = 3$

$|^{-}3| = 3$

Notice the similarity to

$| 3| = 3$

$| 3| = 3$

In general, if n is any whole number, then the absolute values of the integers ^{+}n and ^{-}n are $|^{+}n| = n$, $|^{-}n| = n$

It will be convenient to talk about the direction of an integer just as we have talked about the direction of a directed number. However, the directions will now be called "positive" and "negative". For example:

$^{+}3$ has the positive direction
$^{-}3$ has the negative direction

We now know enough to give instructions for computing the sum of a pair of integers in a way that is exactly analogous to computing the sum of two directed numbers.

If b and c are integers, then

A. The sum b + c has the same direction as the addend having the larger absolute value. (If the addends have the same absolute value, then the direction of either addend may be used.)
B. The absolute value of b + c is
   1. The sum of their absolute values whenever b and c have the same direction.
   2. The difference of their absolute values (the larger – the smaller) whenever b and c have opposite directions.

The flow chart that we made for computing the sum of two directed numbers, b + c, can be used with very slight modification to obtain a flow chart for computing the sum of two integers. What is this slight modification?

4.20 Exercises

1. Construct a flow chart for computing.
   (a) The sum of the integers b and c: b + c
   (b) The difference of the integers b and c: b – c

2. Compute the absolute value of each of the following.

   (a) $^{+}9$ (e) $^{+}7 - {}^{-}3$
   (b) $^{-}9$ (f) $^{-}7 - {}^{+}3$
   (c) $^{+}7 + {}^{-}3$ (g) $^{+}8 + {}^{+}5$
   (d) $^{-}7 + {}^{+}3$ (h) $^{-}8 + {}^{-}5$

3. Considering "absolute value" as a mapping, what is its domain? What is its range?

4. We say that whole number b is greater then whole number c if b – c is a whole number other then zero. Find the solution set for each of the following from the set of integers.

   (a) $|a| + 1 = 2$ (e) $|e| < 2$
   (b) $|b| - 2 = 1$ (f) $|f + {}^{+}3| < 2$
   (c) $|c + {}^{+}4| = 3$ (g) $|^{+}3 - g| < 3$
   (d) $|d + {}^{+}4| = {}^{-}3$ (h) $^{+}2 < |h - {}^{+}3| < {}^{+}5$

5. (a) Obtain 2 rulers. Figure out how they can be used to compute the sum of 3 and 4. Sketch a picture to show how the rulers should be placed.
   (b) Obtain 2 strips of cardboard. Draw a similar scale on each to show integers on a number line.

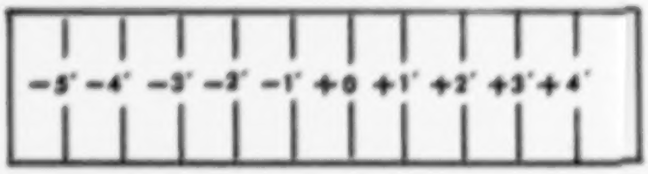
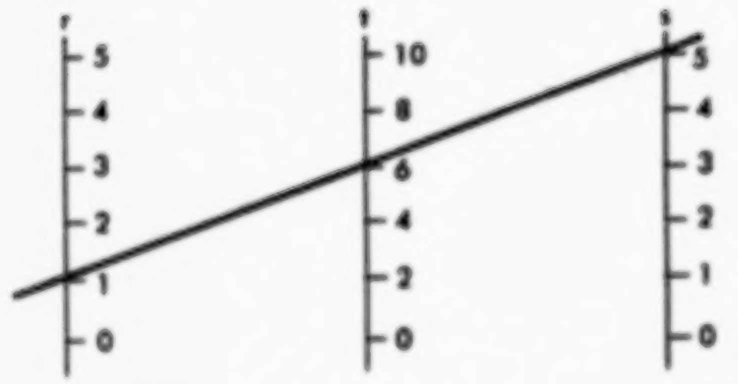


Figure out how your strips may be used to compute the sum of $^{-}5$ and $^{+}3$. Sketch a picture to show how the strips should be placed.

6. The figure on the next page (called a nomogram) consists of 3 parallel lines, equally spaced, with uniform scales as shown. This figure, with the aid of a straight edge, can be used to compute the sum of two whole numbers.



The straight line shows that the sum 1 + 5 is 6.

(a) Draw a line to show how to compute 4 + 2.
(b) Draw a line to show how the nomogram can be used to compute 8 − 5.
(c) Modify and extend this nomogram so that it can be used to compute sums and differences of integers. On your nomogram draw a line for each of the following to show the computation and give the computed value.

(1) $^{+}3 + {}^{+}4$	(4) $^{+}3 - {}^{+}4$
(2) $^{+}3 + {}^{-}4$	(5) $^{+}3 - {}^{-}2$
(3) $^{-}3 + {}^{-}4$	(6) $^{-}3 - {}^{-}5$

4.21 Additive Identity Element and Additive Inverses

Of all the whole numbers which number is the easiest to compute with as an addend? After a little thought you will probably say 0 because there is almost nothing to do. The sum of any number and 0 is that number. Thus, for every whole number n

$$n + 0 = n$$

$$0 + n = n$$

Since 0 seems to do nothing in addition, we often say that 0 is the "do – nothing" element for addition. We also call it the "neutral" element for addition of whole numbers. A more commonly used term for 0 is "identity" element for addition of whole numbers. You might think of 0 as not changing the identity of the other whole number under addition. As you probably know, there is an identity element for addition of integers, and this element is $^{+}0$ or $^{-}0$, as $^{+}0 = {}^{-}0$. In fact, for every integer z

$$z + {}^{+}0 = z \qquad {}^{+}0 + z = z$$

We say that $^{+}0$ is the additive identity element for (Z, +).

We have no difficulty in finding solutions among the integers for equations such as:

$$a + {}^{+}3 = {}^{+}0$$

$$^{-}7 + b = {}^{+}0$$

The solution for the first equation is $^{-}3$ because $^{-}3 + {}^{+}3 = {}^{+}0$. While the solution for the second equation is $^{+}7$ because $^{-}7 + {}^{+}7 = {}^{+}0$. We say that $^{-}3$ is the "additive inverse" of $^{+}3$ and that $^{+}3$ is the "additive inverse" of $^{-}3$. In general, whenever r and s are integers such that

$$r + s = {}^{+}0$$

We say that r and s are additive inverses of each other. We say that r is the additive inverse of s and that s is the additive inverse of r. In fact, we shall often refer to $^{-}3$ as the "additive inverse of $^{+}3$" rather then as "negative 3". This language is especially advantageous when referring to the additive inverse of an integer denoted by, say, "n". We shall use the symbol "^{-}n" to denote the additive inverse of the integer. If n happens to be a negative integer, say $^{-}3$, we see that ^{-}n, in this case, is actually the positive number $^{+}3$. For this reason we shall always refer to ^{-}n as the "additive inverse" of n and not as "negative n". It follows then, for this example, that $-(^{-}3) = {}^{+}3$, and that $-(^{+}3) = {}^{-}3$, for the additive inverse of $^{-}3$ is $^{+}3$ and the additive inverse of $^{+}3$ is $^{-}3$.

4.22 Exercises

1. Find the additive identity element for each of the following systems:

(a) (W, +)	(d) (Z, +)
(b) (W, +)	(e) $(Z_5, +)$
(c) (W, +)	(f) $(Z_6, +)$

2. Find the additive inverse of each of the following integers.

(a) $^{-}73$	(e) $^{+}0$
(b) $^{+}28$	(f) $^{-}7 - {}^{+}3$
(c) $^{-}7 + {}^{+}3$	(g) $^{+}7 - {}^{-}3$
(d) $^{+}7 + {}^{-}3$	(h) $^{-}3 - {}^{+}7$

3. Try to give an argument that there is exactly one
   (a) Additive identity element for (Z, +). Hint: Show that for integers x and c if $x + c = c$ then $x = {}^{+}0$.
   (b) Value for x that fits the following equation (involving only integers)

$$x + a = b$$

Hint: Show that $x = b + {}^{-}a$

(c) Additive inverse of $^{+}3$. Hint: Show that if $x + {}^{+}3 = {}^{+}0$ then $x = {}^{-}3$.

(d) Additive inverse for each integer, n.

4. Suppose that for integers a, b, and c we have $a < b$ and $b < c$. Try to give an argument that
   (a) $a < c$
   (b) $a + c < b + c$
   (c) $^{-}a > {}^{-}c$
5. Find the additive inverse of 3 for these systems:
   (a) $(Z_5, +)$
   (b) $(Z_6, +)$
   (c) $(Z_7, +)$
6. Try to give an argument to prove that for integers, the only value of x that fits the equation
   (a) $x + c = r + c$ is r
   (b) $x - c = r - c$ is r
7. Suppose that a, b, c, d are integers such that

$$a = b \text{ and } c = d$$

Try to prove that

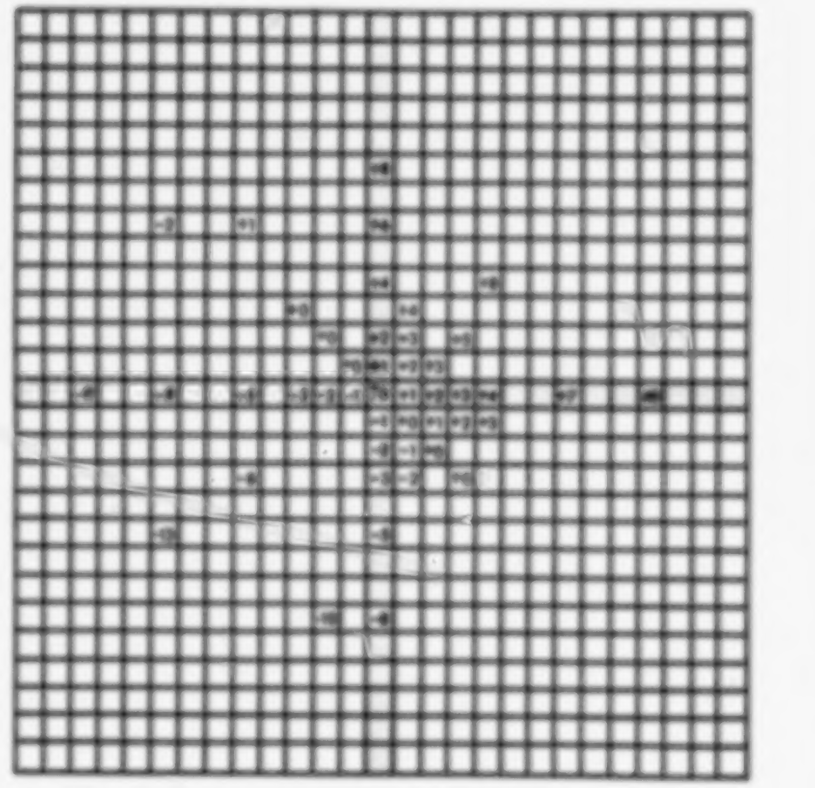
(a) $a + c = b + c$	(e) $a = -(^{-}a)\ a = -(^{-}a)$
(b) $a + c = b + d$	(f) $-(a + b) = {}^{-}a + {}^{-}b$
(c) $a - c = b - c$	(g) $-(a - b) = {}^{-}a + b$
(d) $a - c = b - d$	(h) $a + {}^{-}b = a - b$

(i) $a - {}^{-}b = a + b$

8. The figure at the right can be used as an addition table for integers if properly completed.
   (a) Complete filling in the table as far as you have room and try to figure out how to use it for computing sums.
   (b) Check your table by computing
      1. $^{+}7 + {}^{+}5$
      2. $^{+}7 + {}^{+}5$
      3. $^{-}7 + {}^{+}5$
      4. $^{-}7 + {}^{-}5$
   (c) Try to figure out how this table can be used for computing differences and compute

1. $^{+}7 - {}^{+}5$	4. $^{-}7 - {}^{-}5$
2. $^{+}5 - {}^{+}7$	5. $^{-}5 - {}^{+}7$
3. $^{-}7 - {}^{+}5$	6. $^{-}5 - {}^{-}7$

   (d) What do you notice about all the boxes for which there is the same number (say $^{+}4$)?
   (e) What can you say about the numbers for the boxes along a diagonal running from the upper right corner to the lower left corner?
   (f) How does the fact that $^{+}0$ is the additive identity show up in the table?
   (g) How does the fact that the sum of any pair of integers gives but one sum show up in the table?
   (h) How is commutativity of addition reflected in the table?
   (i) How is associativity of addition reflected in the table?
   (j) Study the table and see what other patterns you can discover. Try to prove that your pattern continues to hold.
   (k) How does the fact that an element has exactly one inverse show up in the table?



9. The successor mapping, S, on the set of integers may be defined as follows:

$$S(^{-}5) = {}^{-}4,\ S(^{+}0) = {}^{+}1,\ S(^{+}3) = {}^{+}4,\ \ldots$$

and in general, for every integer n

$$S(n) = n + {}^{+}1$$

In other words, S(n) is the image of n where

$$S: n \qquad n + (^{+}1)$$

(a) Compute:
   1. $S(^{+}9) + S(^{+}7)$
   2. $S(^{+}9) + {}^{+}7)$
   3. $S(^{+}9) - S(^{+}7)$
   4. $S(^{+}9 - {}^{+}7)$
   5. $S(^{+}10 - {}^{+}3)$

(b) Try to prove that if a and b are integers
   1. $a < S(a)$
   2. $a - b = S(a) - S(b)$
   3. $S(a + b) = S(a) + b$
   4. $S(a) + b = a + S(b)$
   5. $S(a) < S(b)$ if and only if $a < b$
   6. $a < b$ if $S(a) < b$
   7. $S(a)$ need not be less than b if $a < b$

4.23 Summary

1. A directed number defines a translation that associates a point with every point of the plane.
2. Addition of directed numbers may be interpreted as a composition of translations.
3. Addition of directed numbers when considered as a composition of translations is an operation.
4. (W, +) is not a group.
   A mapping between two operational systems is an isomorphism if the mapping is one-to-one and preserves "products". The two systems are then said to be isomorphic.
6. (W, +) and (W, +) are isomorphic.
7. (W, +) is a commutative group.
8. (W, +) and (Z, +) are isomorphic.
9. Subtraction is an operation on (Z, +).
10. Z may be constructed from W through ordered pairs. An integer is now considered to be a set of certain ordered pairs of whole numbers.
11. The integers may be ordered according to greatness. If b–c is strictly positive, then $b > c$.
12. If n is any whole number then the absolute value at ^{+}n and ^{-}n is n: $|^{+}n| = |^{-}n| = n$.
    The absolute value of an integer may also be defined as the difference between the whole numbers in any of its ordered pairs, subtracting the smaller from the larger. The absolute value of $^{+}0$ is 0.

Vertical bars are used to denote absolute value.
$|^{-}3| = ;\ |^{+}3| = 3$

4.24 Review Exercises

1. Compute:

(a) $23 + 58$	(d) $23 - 58$
(b) $23 + 58$	(e) $58 - 23$
(c) $23 + 58$	(f) $28 - 58$

2. Compute:

(a) $^{+}38 + {}^{+}57$	(d) $^{+}38 - {}^{+}57$
(b) $^{+}38 + {}^{-}57$	(e) $^{+}57 - {}^{+}38$
(c) $^{-}38 + {}^{-}57$	(f) $^{-}38 - {}^{+}57$

3. Solve within (Z, +)

(a) $n + {}^{-}7 = {}^{+}2$	(e) $n - {}^{-}7 = {}^{+}2$
(b) $n + {}^{+}7 = {}^{+}2$	(f) $n - {}^{+}7 = {}^{+}2$
(c) $n + {}^{-}7 = {}^{-}2$	(g) $n - {}^{-}7 = {}^{-}2$
(d) $n + {}^{+}7 = {}^{-}2$	(h) $n - {}^{+}7 = {}^{-}2$

4. Solve within (Z, +)
   (a) $^{-}9 + {}^{-}n = {}^{+}1$
   (b) $^{+}9 + {}^{-}x = {}^{-}1$
   (c) $^{-}23 + {}^{+}6 = {}^{-}y + {}^{+}5$
   (d) $^{-}47 - {}^{-}17 = {}^{-}7 + {}^{-}a$
5. Solve within (Z, +)
   (a) $|n| = {}^{+}3$
   (b) $|n| = {}^{-}3$
   (c) $|n - 3| = {}^{+}7$
   (d) $|^{-}n - 3| = {}^{+}7$
   (e) $|3 - n| = {}^{+}7$
   (f) $|3 - {}^{-}n| = {}^{+}7$
   (g) $|n + 3| = {}^{+}7$
   *(h) $|n| + |n + 3| = {}^{+}9$
   (i) $|n| < {}^{+}3$
   (j) $|n - 5| < {}^{+}3$
   (k) $|n - 3| < {}^{+}5$
   (l) $|n| + |n + 3| < {}^{+}5$
6. Insert the symbol that fits: >, = <
   (a) $^{-}37 + {}^{-}29$ $\qquad$ $^{-}70$

(b) ${}^{+}37 + {}^{-}73$	${}^{-}24$
(c) ${}^{-}73 - {}^{+}17$	${}^{+}80$
(d) ${}^{+}67 + {}^{-}75$	${}^{-}12$
(e) $-({}^{-}5)$	${}^{+}5$
(f) $-({}^{+}6)$	${}^{+}6$
(g) $\lvert {}^{-}7 + {}^{-}3 \rvert$	$\lvert {}^{+}7 + {}^{-}3 \rvert$

7. For each statement answer TRUE or FALSE, whichever fits.

(a) Subtraction on $(Z,+)$ is an operation.

(b) $(Z,+)$ is isomorphic to $(W,+)$
For all integers r, s, t

(c) $|r - {}^{+}7| = |{}^{+}7 - r|$

(d) $|r - s| = |s - r|$

(e) $|r - s| = |r + s|$

(f) $r - s = r + {}^{-}s$

(g) $-({}^{-}r) = r$

(h) $-(r - s) = {}^{-}r + s$

(i) $|r - s| < |r + s|$

(j) $|r + s| < |r| + |s|$

PROBABILITY AND STATISTICS

CHAPTER 5

5.1 Introduction

The Fish and Game Commission often must estimate the number of fish in a lake. But, they certainly cannot catch every fish in the lake and count it! Instead, they catch a sample in a net, tag them, and throw them back into the lake. After allowing time for the first sample to mix thoroughly with the fish population, they catch a second sample and count the number of tagged fish in this sample. The fraction of tagged fish in the second sample is an estimate of the fraction of tagged fish in the lake. For example, if the first sample numbers 100 and the second sample 200, of which 50 are tagged, it is assumed that about 50/200 or 1/4 of the fish in the lake are tagged. Only 100 fish were tagged, so 100 is about 1/4 of the fish in the lake.

Question: On the basis of the above estimate, how many fish are in this lake?

A similar estimation problem is often met in industry. For instance, in the manufacture of light bulbs it is important to control the quality of bulbs coming off the assembly line. Since it is not practical to test the burning time of each bulb, a sample of several bulbs is selected and tested. The fraction of defective bulbs in the sample is then used as an estimate of the fraction of defective bulbs in the lot of bulbs being produced. This fraction is called the relative frequency of defective bulbs. If the sample consisted of 50 bulbs, of which 5 were defective, the relative frequency of defective bulbs in the sample would be .1. .1 could be used as an estimate of the relative frequency of defective bulbs in the whole lot.

Today many users of mathematics need the ability to make estimates with a high degree of confidence, in situations where the actual results are uncertain. Important decisions are often based on these estimates.

Question: What are some ways relative frequencies might be used by

(a) The weather bureau;
(b) An auto insurance company;
(c) The National Safety Council;
(d) A life insurance company;
(e) The manager of a supermarket?

5.2 Discussion of an Experiment

The experiment that we discuss here is that of tossing a die. You may think of the experiment as a set of trials and an associated set of outcomes. In this case a trial consists of one toss of the die. The set of outcomes is pictured below:

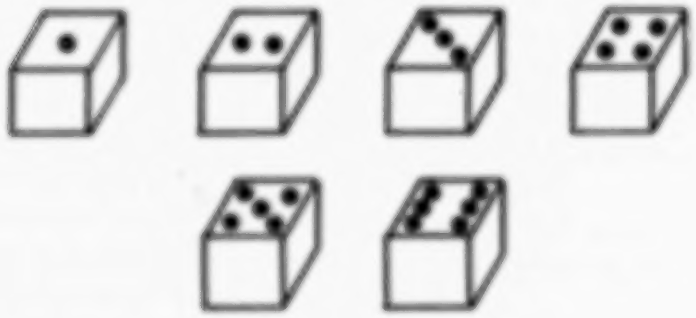


Figure 5.1

We say that the outcome is 2 if the die comes to rest with two dots on the "up" face. The outcome set is {1, 2, 3, 4, 5, 6}. For each trial the outcome is the number of dots on the up face. If a trial results in a certain outcome, we say that this outcome occurs.

Any subset of a set of outcomes is called an event. Thus, {2,4,6} can be described as the event that the outcome is an even number. An event is said to occur if any one of its outcomes occurs.

We can simplify the description of event {2,4,6} by letting {2,4,6} = K. Then if an outcome is an even number, we say that K occured. For example, if the outcome of a trial was 2 we say that K occurred.

Since any subset of an outcome set is an event, the outcome set {1, 2, 3, 4, 5, 6} itself is an event. It could be described as the event that the outcome was a whole number between zero and seven. A subset containing a single outcome is called a simple event. For example, in this experiment {2} is a simple event.

Below is a table showing the results of an experiment that was performed. The experiment consisted of rolling a die 24 times with the outcome set {1, 2, 3, 4, 5, 6}. The first column of the table shows the outcomes, the second shows the tally of the occurrences of each outcome; the third shows the frequency or number of occurrences of each outcome; the fourth shows the relative frequency of each outcome.

Table 1 24 Tosses of a Die

Outcomes	Tally	Frequency	Relative Frequency
1	///	3	3/24 = 1/8
2	𝍸	5	5/24
3	////	4	4/24 = 1/6
4	//	2	2/24 = 1/12
5	𝍸	5	5/24
6	𝍸	5	5/24

5.3 Exercises

1. Tabulate the following events of the die tossing experiment. That is, list all outcomes that satisfy the condition.

(a) The outcome is less than 3. Ans. {1,2}.
(b) The outcome is greater than 5.
(c) The outcome is less than 3 or greater than 5. Where "or" is used tabulate all outcomes that satisfy at least one of the two conditions.
(d) The outcome is greater than 1 and less than 4. Where "and" is used, tabulate only outcomes that satisfy both conditions.
(e) The outcome is greater than 2 and less than 3.
(f) The outcome is a member of the outcome set.
(g) The outcome is a prime number.

2. Describe the following tabulated events.

(a) {2, 4, 6}
(b) {1,3,5}
(c) {1, 6}
(d) {2, 3, 5}

3. Perform the experiment of tossing a die 24 times and record the results in a table like Table 1. Compare your results with those in Table 1.

(a) Which of your frequencies were the same as those in Table 1?
(b) Which of your relative frequencies were the same?
(c) Do you think that you will always get the same relative frequencies in repeating this experiment? If you have doubts, try it!
(d) Add the relative frequencies in the last column of your table. Add the relative frequencies in Table 1. Were the sums the same?
(e) Find out what the other students in your class found as the sum of the relative frequencies in their table.
(f) If you all found the same sum, try to explain why this happened.

4. Suppose that you had a coin with a "head" on both sides and performed the experiment of tossing this coin 100 times.

(a) What would be the frequency of the outcome "heads-up"?
(b) What would be the relative frequency of this outcome?
(c) Would you say that the outcome, heads, was certain?
(d) What is the relative frequency of any event that is certain?

5. In the same experiment of tossing the "two-headed" coin:

(a) What is the frequency of the outcome "tails"?
(b) What is the relative frequency of this outcome?
(c) If "heads" was a certain event for this experiment, how would you describe the event, "tails"?

6. What is the relative frequency of an event that is impossible?

Use Table 1 for exercises 7, 8, and 9.

7. What is the relative frequency of 2? Of 4? Of 6?

8. What is the sum of the relative frequencies in exercise 7?

9. What is the relative frequency of the event that the outcome is an even number?

10. What conjecture might you make on the basis of the answers to exercises 7, 8, and 9?

11. It is interesting to find out what happens to relative frequencies as you increase the number of trials. Instead of repeating an experiment many times, you may save time by combining your results with those of the other students in the class.

Use the results for the event {5} in your die-tossing experiment (exercise 3) for the following experiment. First, draw this chart on the chalkboard:

Cumulative Number of Trials	Cumulative Frequency	Relative Frequency
24		
48		
72		
96		
etc.		

(a) Have the first student go to the board and enter his frequency and relative frequency for the outcome 5 in the columns to the right of 24.
(b) Have the next student go to the board, add his frequency for the same event to the frequency of the first student, and enter the sum in the second row of the cumulative frequency column. He then divides this sum by 48, and enters the quotient (in fraction-form) in the relative frequency column.
(c) The third student follows the same procedure in the third row and so on.
(d) If the first three students had 3, 4, and 5 respectively, for the frequency of the outcome 5, the entries would look like this:

24	3	3/24 or 1/8
48	7	7/48
72	12	12/72 or 1/6

Question: If you have 20 students in your class, how many trials will you have by the time each student has recorded his results on the chart?

(f) Experience indicates that as you increase the number of trials in an experiment to very large numbers, the relative frequencies of an event tend to vary less and less from some specific number.

(g) Even though your class project does not involve very large numbers, compute the differences of consecutive pairs of relative frequencies to see if they tend to decrease. (See the illustration below for a suggestion on how to proceed).

Number of trials	Relative Frequencies	Consecutive Differences
24	a	
		$\|a-b\|$
48	b	
		$\|b-c\|$
72	c	
		$\|c-d\|$
96	d	
etc.	etc.	etc.

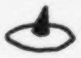
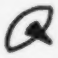
(h) The property discussed in this exercise is called the stability of relative frequencies.

The following statements summarize the ideas of the preceding exercises.

1. The relative frequency of an event is:
   (a) 0, 1 or a number between 0 and 1;
   (b) 1 if the event is certain;
   (c) 0 if the event is impossible;
   (d) the sum of the relative frequencies of its simple outcomes.
2. The sum of the relative frequencies of the outcomes in an experiment is 1.
3. Relative frequency has the property of stability. This idea will be explored further in experiments and illustrated graphically.

5.4 Experiments to be Performed by Students

For this experiment students should work in groups of two or three, but each one should perform the experiments while his teammates help him tally the results. In this way you can do experiments where you need a large number of trials but want to use the same experimental object such as the same die, coin or thumbtack. The large number of trials can be achieved by combining the results of the three members on a team.

5.4.1 Experiment: 20 tosses of a thumbtack repeated 5 times. Toss a thumbtack on a hard surface where it will bounce before coming to rest. (We hope that this will take all of the prejudice out of the way you toss.) The simple outcomes will be "Up" and "Down."  Up  Down

Each student should make a chart like the one below and tabulate his results. When each student on a team has completed 5 groups of 20 tosses, the three students working together should fill in Table 2 for the cumulative results, using the outcome UP. (Is information lost by only considering the outcome UP? Why or why not?)

5.4.2 Tables

Twenty Tosses of a Thumbtack Repeated Five Times

	UP			DOWN		
Trials	Tally	Frequency	Rel.Freq.	Tally	Frequency	Rel.Freq.
1st 20	eg ~~////~~ ~~////~~ //	eg 12	$\frac{12}{20}=\frac{3}{5}$ eg	eg ~~////~~ ///	eg 8	$\frac{8}{20}=\frac{2}{5}$ eg
2nd 20	~~////~~ ////	9	$\frac{9}{20}$	~~////~~ ~~////~~ /	11	$\frac{11}{20}$
3rd 20						
4th 20						
5th 20						

Table 2

Cumulative Results for 300 Tosses of a Thumbtack
(groups of twenty)

Cumulative Number of Trials	Cumulative Frequency for UP	Relative Frequency for UP	Consecutive Differences
20	e.g. 12	e.g. $\frac{12}{20}=\frac{3}{5}=.6$	
40	21	$\frac{21}{40}\approx .53$	$\frac{3}{40}$
60	27	$\frac{27}{60}=.45$	$\frac{9}{120}$
80	etc.		
100			
120			
140			
160			
180			
200			
220			
240			
260			
280			
300			

Table 3

5.4.3 Graphs

The best way to illustrate the stability of relative frequency is through the use of graphs. Each student will make two graphs to show the results of the thumbtack experiment. The relative frequencies for UP, tabulated in tables 2 and 3 will be used.

The first graph will show the relative frequencies for UP for each group of twenty tosses. The second graph will show the relative frequencies for UP for increasing numbers of trials. The two graphs below illustrate the procedure using results of an imaginary experiment:

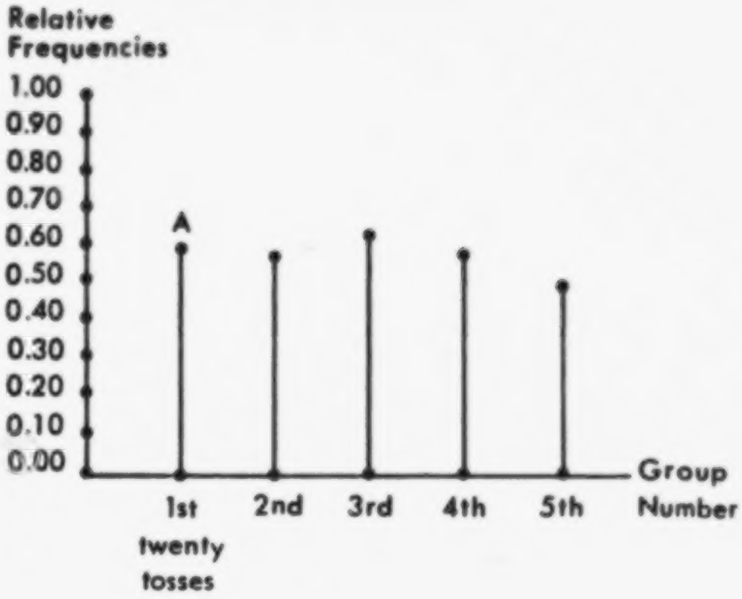


Table 4

This table shows that for the five groups of tosses the relative frequency did not vary much. Point A shows that in the first group of twenty tosses the relative frequency of UP was .6. For the five groups illustrated in the graph, the greatest relative frequency was about 0.62 and the least about 0.48. The difference between the greatest and least is 0.14.

Now construct a graph similar to that in Table 4 using the results of your experiment tabulated in Table 2.

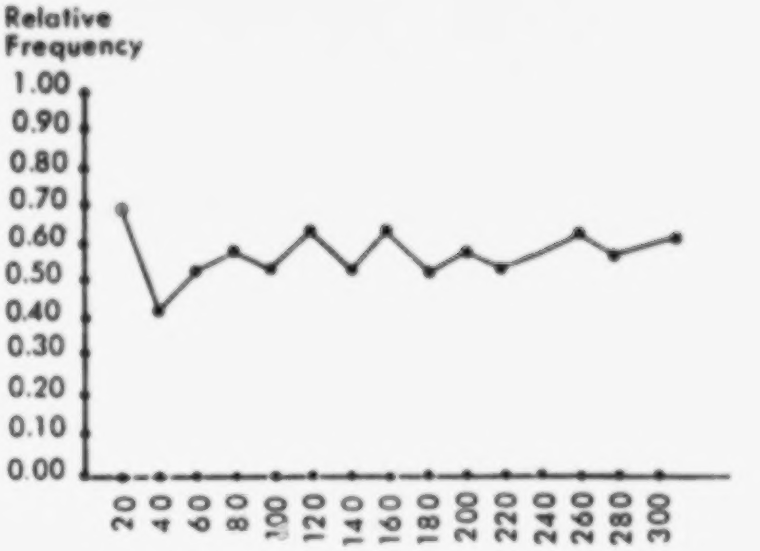


Table 5

This table shows that as the number of tosses increased in this particular experiment, the relative frequencies did indeed "stabilize" around a number (about 0.62). With a different thumbtack, the number might have been different. Now construct a graph similar to that in Table 5 using the results of your team tabulated in Table 3. Do your relative frequencies tend to stabilize around a number? Is this number near .62? If not, can you explain the difference? Compare your cumulative relative frequency with those of other teams.

Question: What do you think the results might be with a thumbtack that has a very small head and a long pin?

5.5 The Probability of an Event

The thumbtack experiment provided us with one example of the tendency of relative frequencies to "stabilize" as the number of trials increases. This tendency is sometimes called the law of large numbers. This law can be verified by many types of experiments.

Our findings about the stability of relative frequencies suggest that we might be able to predict relative frequencies in some cases where they can't be observed or where it would be very impractical to observe them. For example, if you were manufacturing firecrackers, you wouldn't want to test the quality of your product by exploding each one (or maybe you would!). The prediction of relative frequencies is an assignment of numbers to events. The number is called the probability of the event. If you like shorthand, you may use the symbol "P(E)" to stand for the "probability of the event E".

All rights involve responsibilities, and the right to assign probabilities to events obligates us to obey certain laws. Suppose you feel, on the basis of experience, that one of your coins will come up heads about 1/3 of the time. You decide to assign 1/3 to P(H) (The probability of heads). What must you then assign to P(T)? In other words, about how often would you expect tails?

In short, since probabilities are predictions of relative frequencies we must expect them to obey all of the properties that we have developed for relative frequencies.

1. $0 \le P(E) \le 1$
2. $P(E) = 1$, if E is certain to occur.
3. $P(E) = 0$, if E cannot occur.
4. The sum of the probabilities of the outcomes in an outcome set is 1.
5. P(E) is equal to the sum of the probabilities of the simple outcomes in E.

5.6 A Game of Chance.

Play the following game with another student in your class and decide if it is fair or unfair. Toss a pair of dice (or wooden cubes with numerals from "1" to "6" on the faces if anyone objects to dice) and observe the sum of the outcomes.

Player A gets one point if the sum is 2, 3, 4, 10, 11 or 12. Player B gets one point if the sum is 5, 6, 7, 8, or 9. Notice that there are 6 sums that will give player A a point and only 5 sums that will give player B a point. The first person to get 10 points wins the game.

(a) Pick a partner and play the game 4 times.
(b) How often did player A win? Player B?
(c) Is the game fair? If not, who had the advantage?
(d) If one player has the advantage try to discover why?

5.7 Equally Probable Outcomes

You have seen that we can assign probabilities to the simple outcomes of an experiment on the basis of experience with relative frequencies. We can also make assumptions about the probabilities of the simple outcomes in an outcome set as long as they are consistent with the properties that we developed in connection with relative frequencies.

It is often reasonable to assign the same probability to each of the simple outcomes in an experiment. For example, in tossing a coin, we often assign equal probabilities to heads and tails.

Question: In this case, what is the probability of heads? Tails?

In tossing a die we often assign the same probabilities to 1, 2, 3, 4, 5, and 6.

Question: In this case what is the probability of each outcome?

Question: If there are n equally probable outcomes in an outcome set, what is the probability of each?

If we say that a coin or a die is fair, we mean that all elements of the outcome set have the same probabilities.

If we toss a fair die, what is the probability of the event that the outcome is greater than 4? There are 2 simple outcomes in this event, $\{5, 6\}$. $P(5) = 1/6$ and $P(6) = 1/6$. Since the probability of an event is the sum of the probabilities of its simple outcomes, $P(\{5, 6\}) = 1/6 + 1/6 = 1/3$.

Notice that we could have saved ourselves some trouble by reasoning that since there are two simple outcomes in the event, and six simple outcomes in the outcome set, the probability of the event is 2/6 or 1/3. In general, whenever the simple outcomes are equally probable, the probability of an event is $\frac{n(E)}{n(S)}$, where n (E) stands for the number of outcomes in E and n(S) stands for the number of outcomes in the outcome set.

When selection of members from a set is made so that all possible choices are equally likely, we say that we are selecting a member at random. Consider the experiment of selecting a letter of the alphabet at random. Each letter is equally likely to be chosen. Let V be the event that a vowel is selected, C be the event that a consonant is selected, and A the event that a letter in the alphabet is selected.

Questions: What is P(V)? What is P(C)?
What is P(A)?
What is P(V) + P(C)?

5.8 Exercises

1. Toss a pair of dice of different color, for example one white and the other blue. The outcomes occur in ordered pairs (W,B). There are 6 outcomes for the white die and for each of these there are 6 outcomes for the blue die.

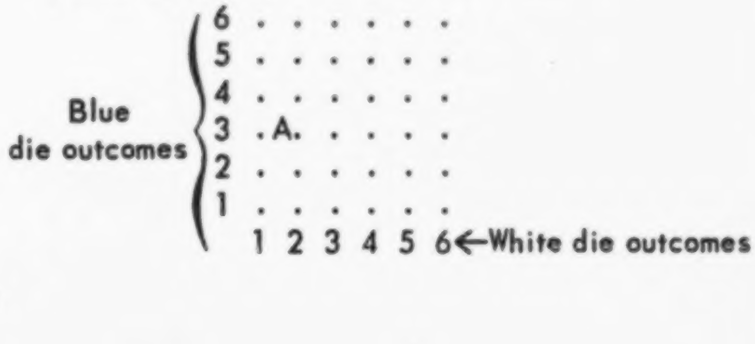
   Question: How many ordered pairs of outcomes are there for the two dice? Use the order (white, blue).

You could record the outcome set in a square pattern as follows:

(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)

(a) Copy the above diagram of the outcome space.
(b) Use a ruler to draw a red line through all of the pairs for which the sum is 7. Do the same for sums of 5, 6, 8, and 9.
(c) Now draw a green line through all the pairs for which the sum is 10. Do the same for sums of 2, 3, 4, 11, and 12.
(d) Let each outcome in the diagram represent a point. How many points are on green lines?
(e) How many points are on red lines?
(f) How many points are in the total outcome set?
(g) If you select a point at random what is the probability that it will be on a green line? a red line?
(h) What is the probability that when you toss a pair of dice, the sum of the outcomes for each die will be 5, 6, 7, 8, or 9? 1, 2, 3, 10, 11 or 12?
(i) Now look back at the dice game of Section 5.4. Was it a fiar game?

2. (a) Using your diagram from Exercise 1 draw a closed curve around the set of points for the event (white die outcome less than blue die outcome) and call this event A.
   (b) Repeat the directions in (a) for event (white die outcome greater than blue die outcome) and call this B.
   (c) Let C be the event that A occurs or B occurs.
   (d) What is P(A)? P(B)? P(C)? P(A) + P(B)? Note that P(A) + P(B) = P(C) and that A and B had no outcomes (or points) in common.
3. Make another diagram of the outcome set but this time, to simplify matters, use dots for the points as below:



Point A in the diagram is associated with (2, 3). To avoid confusion between single outcomes and pairs of outcomes we will call 2 the first coordinate of point A and 3 the second coordinate of A.

(a) Draw a line through the points with equal first and second coordinates. Call the set of points on this line event R.
(b) Draw a line through the points with coordinate sum 8. Call the set of points on this line event T.
(c) Do R and T have any points in common? If so, what are the coordinates of this point? Call the set with only this common point event Q.
(d) What is $P(R)$? What is $P(T)$? What is $P(Q)$?
(e) Let K stand for the event that "R occurs or T occurs". What is $P(K)$? What is $P(R) + P(T)$? Is $P(K) = P(R) + P(T)$?
(g) Does $P(K) = P(R) + P(T) - P(Q)$?
(h) Compare the results of this exercise with exercise 2 and try to discover why in exercise 2, $P(A) + P(B) = P(C)$ and in exercise 3, $P(K) = P(R) + P(T) - P(Q)$.

4. It is a well-known fact that the probability of a newborn child being a girl is about 1/2. What chance does this leave for boys?
   (a) What do you think the probability might be of a family having Boy-Girl-Boy (BGB) in that order?
   (b) The outcome set for the event of having three children is:
   {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}
   (c) How many triples are in the above outcome set?
   (d) Assuming all outcomes to be equally probable, what is the probability of each?
   (e) How many of the triples tabulated in (b) have G as the second letter?
   (f) What is the probability that the second child is a girl?
   (g) Suppose that we change our outcome set to include only those outcomes where we know that the first child was a boy.
   {BBB, BBG, BGB, BGG}
   How many outcomes are in this set?
   (h) What is the probability, using the outcome set of (g) that the second child is a girl?
   (i) In questions (f) and (h), the answers should have been the same. In other words, the fact that the first child was a boy did not influence the likelihood that the second was a girl.

5.9 Another Kind of Mapping

In Chapter 3 you studied mappings from one set of numbers onto another set of numbers and mappings from one set of points onto another set of points. Below is a diagram that portrays a mapping from a set of outcomes onto a set of numbers:

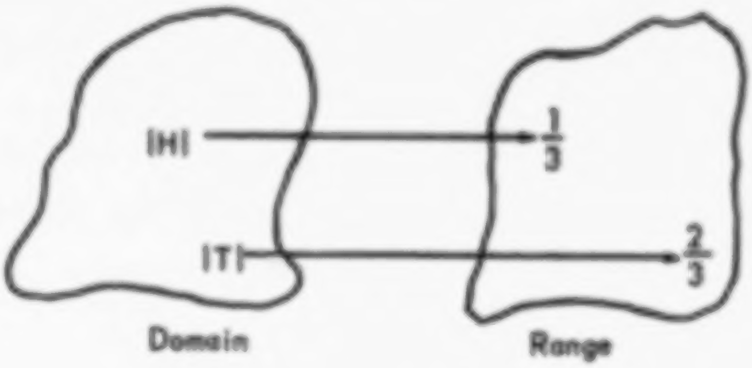


Figure 5.2

Notice that the outcomes seem to be those resulting from the toss of a coin. The images in the range could be the probabilities of the corresponding members of the domain. Is the coin a fair coin? Are the images between 0 and 1? If so, is the sum of the images equal to 1?

The mapping illustrated below shows the probabilities for certain events in a three-child family:

Domain	Range
{Exactly three boys}	1/8
{Exactly two boys}	3/8
{Exactly one boy}	3/8
{No boys}	1/8
Sum	1

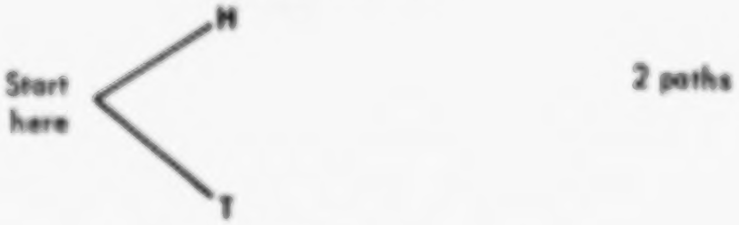
Question: Why is the probability of exactly two boys 3 times as great as the probability of exactly three boys?

Question: Make up an outcome set with 16 outcomes for the children in a four-child family. One of the outcomes, for example, will be BBGB. Illustrate the mapping of the outcomes, exactly four boys, exactly three boys, etc. onto their probabilities.

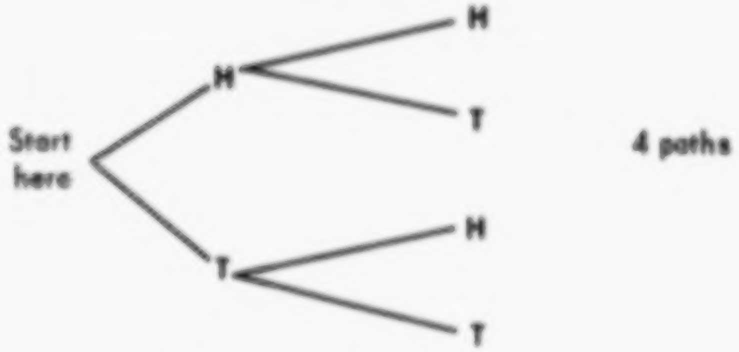
5.10 Counting with Trees

If an experiment involves several objects each having several alternatives, it is often a complicated task to count all of the possible outcomes and identify them. Below are some tree diagrams for coin tossing experiments. If you follow every path in the tree for an experiment you will discover all possible outcomes:

1. Tossing one coin



2. Tossing 2 coins



3. Tossing 3 coins

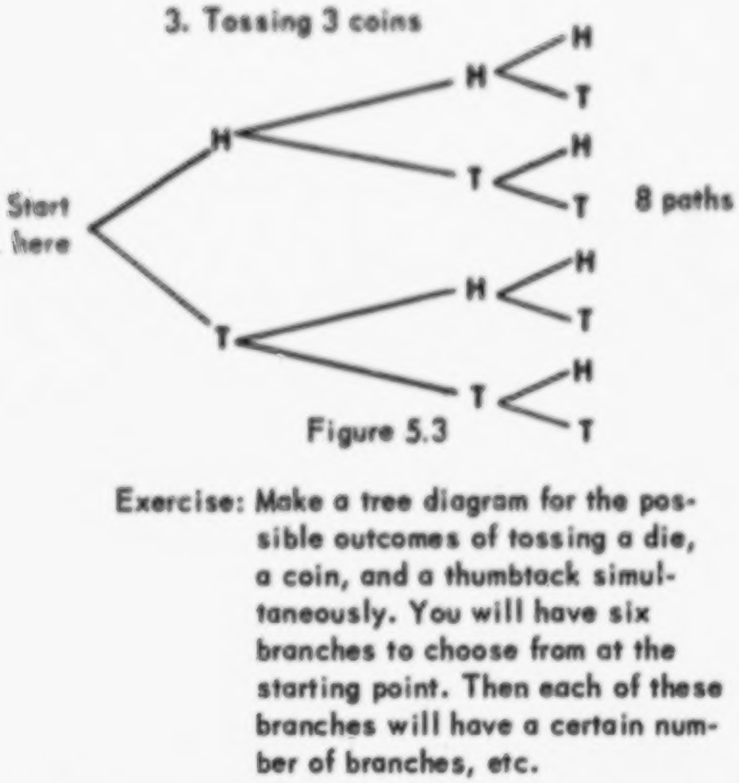


Figure 5.3

Exercise: Make a tree diagram for the possible outcomes of tossing a die, a coin, and a thumbtack simultaneously. You will have six branches to choose from at the starting point. Then each of these branches will have a certain number of branches, etc.

5.11 Summary

The following ideas, which were illustrated in some of the preceding exercises, will be developed in more detail in your later study of mathematics:

1. If two events, A and B, have no members in common, then the probability that at least one of them occurs is the sum of the probabilities of the two events:

   P(A occurs or B occurs) = P(A) + P(B)

2. If two events C and D, have members in common, then the probability that at least one of them occurs is the sum of the probabilities of the two events minus the probability that both occur:

   P(C occurs or B occurs) = P(A) + P(B) −
   P(A occurs and B occurs)

5.12 Research Problems

1. In the diagram below the circles are called 'states' and the routes for legally getting from one state to another are called 'paths.' The numerals in circles A,B,C,D, and E indicate the number of paths from the start to the respective states.

1.

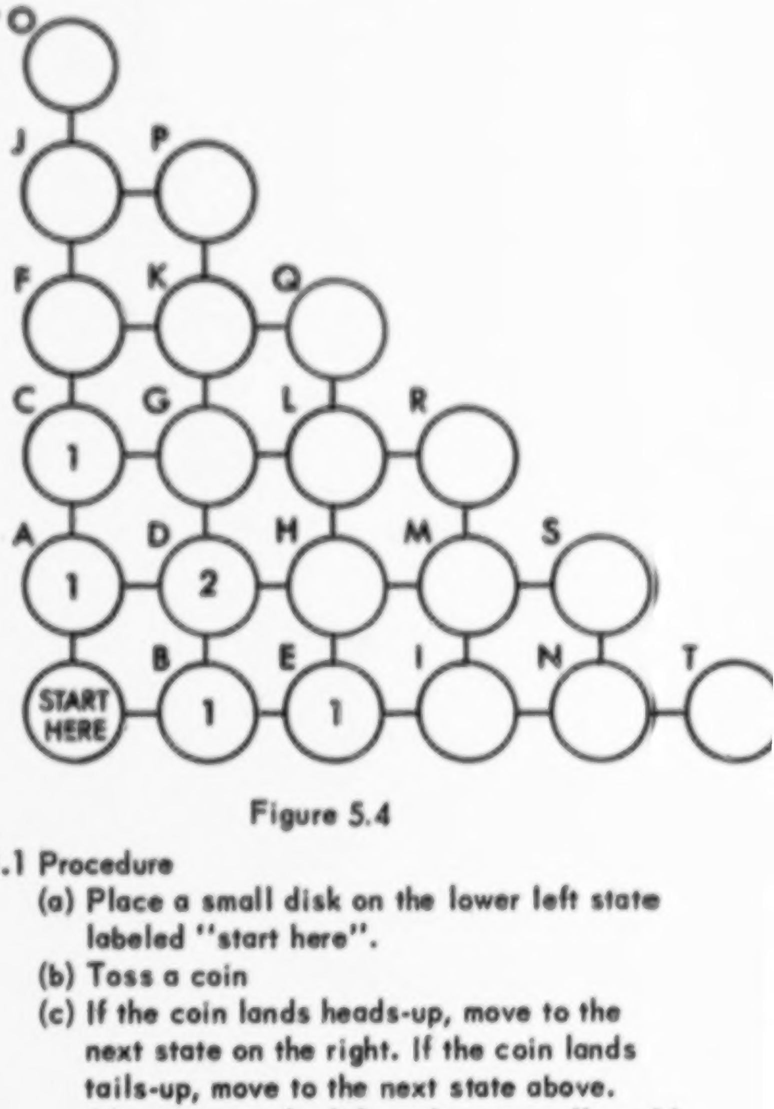


Figure 5.4

1.1 Procedure
   (a) Place a small disk on the lower left state labeled "start here".
   (b) Toss a coin
   (c) If the coin lands heads-up, move to the next state on the right. If the coin lands tails-up, move to the next state above. (No moves to the left or down are allowed.)

1.2 Experiment
   (a) Toss a coin five times and make the proper moves on each toss. What state did you reach?
   (b) Repeat the five-toss experiment 64 times and each time record your destination.
   (c) What was the relative frequency for each destination?

(d) What do you notice about the location of your destinations?

1.3 Experiment

(a) Record your destinations for a two-toss experiment with 32 repetitions.

(b) What was the relative frequency of each destination?

(c) What do you notice about the location of these destinations?

1.4 Counting Paths

(a) Using the rules of our game, there is only one path to each of A,B,C and E but there are two paths to D. State G would have 3 paths, A-C-G; A-D-G; and B-D-G. Make a copy of the diagram of states and record the number of legal paths to each state inside the corresponding circles in the diagram.

(b) Except for the border states in the left column and the bottom row, each state has exactly two possible predecessors, the one below and the one to the left. Find a method of computing the number of paths to a state by using the number of paths to each predecessor.

(c) There are 2 one-toss paths, A and B. There are 4 two-toss paths, A-C, A-D, B-D and B-E. How many three-toss paths are there? Four-toss?

(d) There are 32 five toss paths and 10 of these go to state Q. What is the probability of arriving at Q in five tosses, if we assume each path to be equally probable?

(e) Compute the probabilities for each state in the diagram.

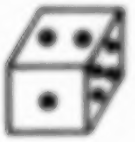
2. The Birthday Anniversary Problem

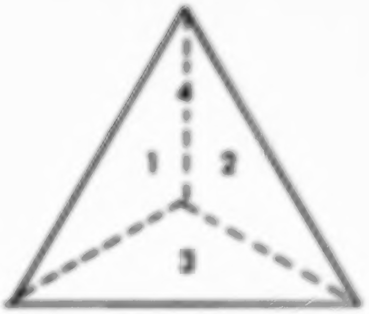
How large a group of people would you need so that the chances would be 50-50 that at least one pair of people in the group have the same birthday anniversary? (Any person born on February 29 will not be considered in this problem.) problem.)

(c) Consult Who's Who in America or a similar book and pick ten samples of 20 people in alphabetical order. Be sure to avoid overlap in your samples. This is then random enough for our purpose. How many of the ten samples contain a pair of people with the same birthday anniversary? Record the relative frequency of this occurence.

5.13 Exercises

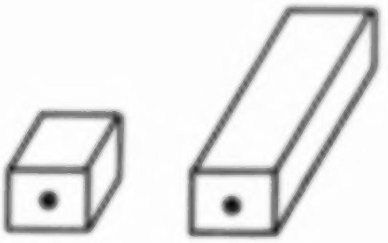
1. (a)





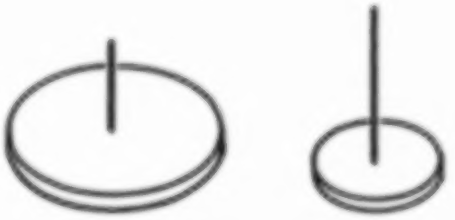
The tetrahedron has four faces. Imagine that each face has a numeral from 1 to 4 respectively. Will the probability of the outcome, 4, be greater for tossing the die or the tetrahedron? What are the probabilities in these two cases?

(b)



Which of the above dice would give the greater probability to the outcome, 1?

(c)

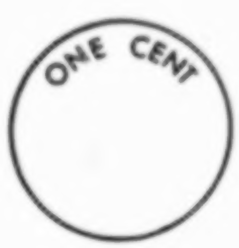


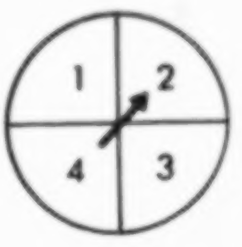
Which tack pictured above would be more likely to come to rest, pin-up? Why?

2. Make up an outcome set, using the simplest outcomes, for each of the following experiments:

(a) Toss a die and a tetrahedron, as in 1. (a), simultaneously. {(1,1), ...}

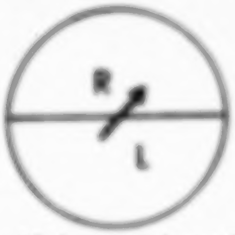
(b) Toss a coin, spin the dial, select a vowel at random.



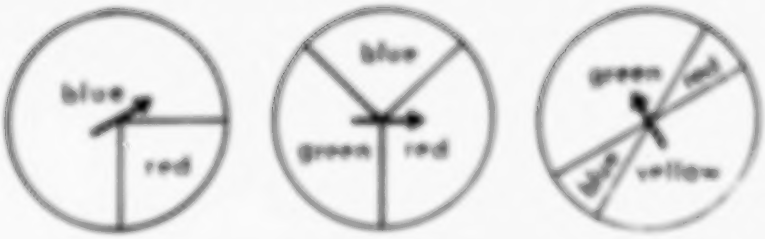


Think of each trial having a triple outcome, such as (H,3,u).

3. $\dot{-3} \quad \dot{-2} \quad \dot{-1} \quad \dot{0} \quad \dot{+1} \quad \dot{+2} \quad \dot{+3}$
   (a) Copy the above diagram and place a disk on the point labeled "0".



   (b) Spin the dial. If the result is R, move the disk to the next point on the right. If the result is L, move the disk to the next integer on the left.
   (c) Repeat moves until you reach +3 or −3. The outcome is whichever of +3 or −3 you reach first.
   (d) Play the game several times and find the relative frequency of +3.
   (e) In what way does the dial affect the relative frequency of +3?
4. A bag contains 3 yellow marbles and 5 green marbles.
   (a) If you select a marble without looking in the bag, what is the probability of selecting a yellow marble? a green marble?
   (b) If you select a yellow marble on the first draw, and do not replace it, what is the probability of drawing a green marble on the second draw?
5. Try to explain the meaning of the probabilities in the following situations:
   (a) An engineer says: The probability that the lamps we manufacture will burn more than 1000 hours is 0.05.
   (b) According to Laplace (1749-1827), a famous French mathematician, the probability that a baby will be a girl is 22/43.
   (c) When you toss two dice the probability that you will get the sum 7 is equal to .17.
   (d) A mathematician who has been consulted concerning inventory problems in a supermarket says: The probability that more than 1000 units of this kind will be sold during a day is 0.1.
   (e) A meteorologist says: When the weather conditions are what they are today, the probability that it will rain tomorrow is .15.
6.



   (a) For which of the above dials is the probability of the spinner stopping on red the greatest?
   (b) Estimate what you think the probability of the event, "red" might be for each of the dials.
7. Make a tree diagram for exercise 2b.
8. In a family with 6 children, what is the probability that all children are boys?
9. Toss a pair of dice of different colors (green and red).
   (a) What is the probability that at least one die will show 1 on the up-face?
   (b) Draw a rectangular diagram of the 36 point outcome set and enclose the points for the event described in (a).
   (c) What is the probability of the event, green die 1 and red die 1?
10. What is the probability that two people selected at random will both have birthday anniversaries on a Wednesday in 1966?

5.14 Statistics

"Seventy-five per cent of the automobile accidents in this state happen within twenty miles of home!"

Statements of this type, in which statistics are presumably used, are often made and often misinterpreted. A fragment of information, such as that mentioned above, leaves many important questions unanswered.

What is the source of this information? Were the accidents only those for which insurance claims were involved? Were they the accidents recorded in police records? Was the information acquired from some sample of accidents, or did it really include every actual accident? Over what period of time did these accidents occur? If the information was based on a sample, how was the sample chosen? What conclusions should we draw? Is it really more dangerous to drive near home, or is it possible that seventy-five percent of all driving is done within twenty miles of home?

The above questions are related to the work done by statisticians. The statistician makes a science of gathering information, organizing it, analyzing it to see if there are any patterns, presenting it in the manner that will be most informative, making predictions on the basis of it, and verifying these predictions by continuing to gather information.

In this section we will illustrate some ways of presenting information about events of various kinds and ask you to gather certain data and present it in tables and graphs.

5.14.1 Presenting Data in Rectangular, Circular, and Bar Graphs

A seventh grade mathematics class made a survey of their junior high school to find out the proportion of the student body that used various methods of travel to school. First they recorded the results in a table, then they made a rectangular graph, a circular graph, and a bar graph to present the data that they gathered.

TABLE 6 Methods of Transportation to School Used by Students in a Junior High School

	Number of Students	Per cent
Walk	631	43
Automobile	220	15
Bus	455	31
Bicycle	161	11
Total	1467	100

TABLE 7 Rectangular Graph

Methods of Transportation to School Used by Students in a Junior High School

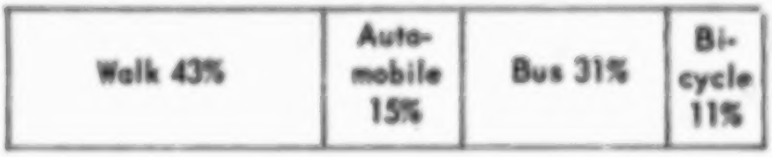


TABLE 8 Circular Graph

Methods of Transportation to School Used by Students in a Junior High School

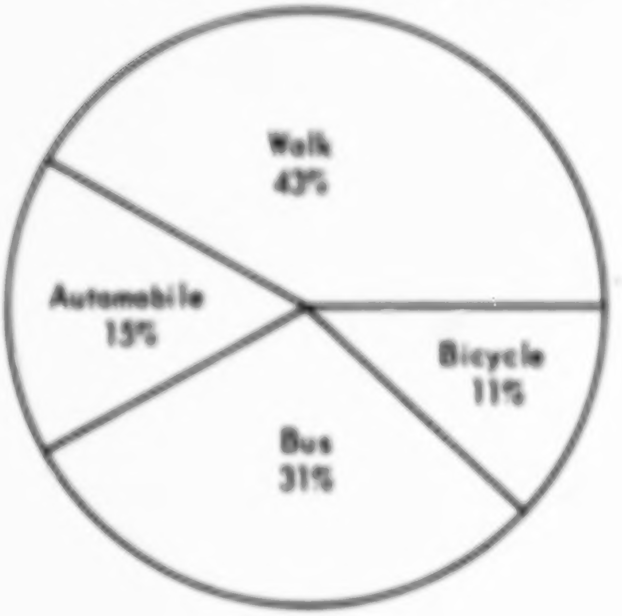
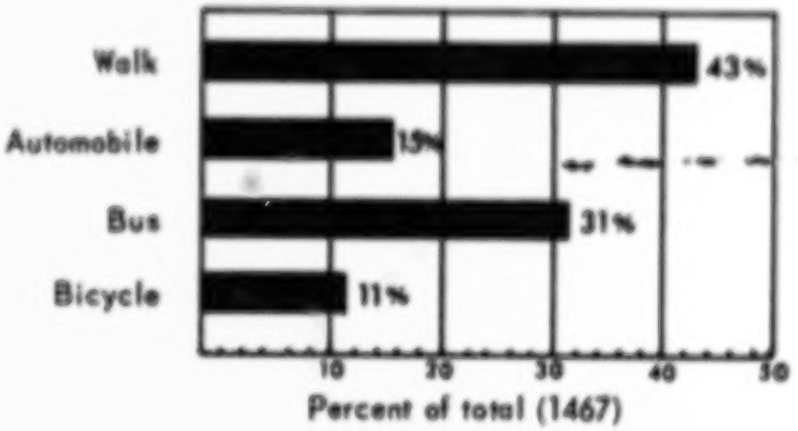


TABLE 9 Bar Graph

Methods of Transportation to School Used by Students in a Junior High School

Method of Transportation to School Used by Students in a Junior High School



5.14.2 Exercises

1. (a) About how many times as many students came by bus as by automobile?
   (b) About how many times as many students walked as came by bicycle?
   (c) What means of travel was used by the least number of students? the greatest number?

(d) Which type of graph was the most effective in presenting least and greatest percentages? for comparing the percentages?

2. (a) Obtain information from members of your class on their means of travel to school.
   (b) Gather data from your class on, 1) how many go home for lunch, 2) how many bring their lunch, 3) how many purchase their lunch in the school cafeteria, and 4) how many are in none of the three preceding groups (record this as "other").
3. Present each set of data tabulated in Exercise two by means of a graph. Use a rectangular, circular or bar graph.

5.14.3 Presenting Data in Tables

During the summer playground program, the children engaged in many activities, including basketball foul-shooting. Near the end of the program, the director organized a foul-shooting contest. A group of twenty boys and a group of twenty girls were selected as the first to participate. Each one had ten tries and the results were tabulated as below:

TABLE 10 Number of Baskets out of Ten Tries in a Foul-shooting Contest

GIRLS

Contestant	Score	Contestant	Score
1	4	11	7
2	10	12	1
3	6	13	5
4	8	14	3
5	2	15	2
6	8	16	7
7	9	17	8
8	1	18	8
9	8	19	8
10	4	20	9

BOYS

Contestant	Score	Contestant	Score
1	1	11	7
2	3	12	8
3	3	13	1
4	5	14	3
5	9	15	1
6	7	16	7
7	9	17	10
8	7	18	6
9	6	19	6
10	9	20	10

The scores in the above table occur in the same order as that in which the players participated. As you look over the scores, try to answer the following questions?

1) Did the girls do better that the boys?
2) What is a good guess for the girls' average? boys' average?
3) What would you estimate as the middle score for the girls? for the boys?
4) What score occurred most frequently for the girls? the boys?
5) How were the scores distributed? That is, were most of the scores either very high or very low; or did most of them cluster somewhere in between?
6) In the table below, the same scores are ranked by size.

Now try to answer the same questions for the table below.

TABLE 11 Number of Baskets in Ten Tries in a Foul-shooting Contest

GIRLS	BOYS
1	1
1	1
2	1
2	3
3	3
4	3
4	5
5	6
6	6
7	6
7	7
8	7
8	7
8	7
8	8
8	9
8	9
9	9
9	10
10	10

Notice that Table 11 certainly gives more information about the middle score and the scores that would occur at about the 1/4 mark and 3/4 mark. You also get the feeling that neither group was unquestionably superior to the other.

The next table shows the frequency of scores grouped by intervals. This type of table is particularly effective when the data consist of large numbers of measures such as weights, lengths or time intervals.

TABLE 12 Number of Baskets in Ten Tries in a Foul-shooting Contest (Scores Grouped into Five Intervals)

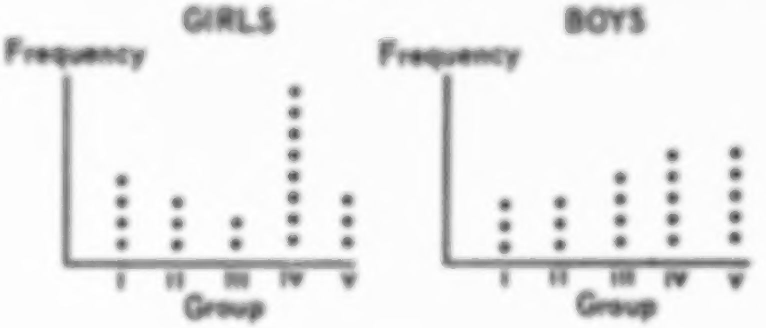
Group	Class Interval	Frequency	
		Girls	Boys
I	0 - 2	4	3
II	2 - 4	3	3
III	4 - 6	2	4
IV	6 - 8	8	5
V	8 - 10	3	5

Each interval includes the greater number. e.g. 0-2 includes 2.

Below are two dot frequency diagrams for the same information represented in the tables 10, 11 and 12:

TABLE 13

DOT FREQUENCY DIAGRAM



5.14.4 Exercises

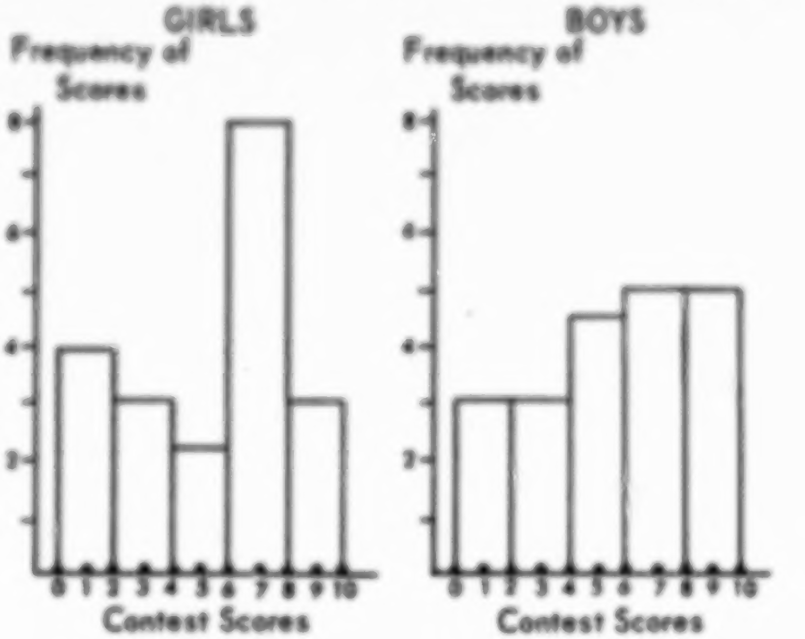
1. Discuss the following:
   (a) The advantage of ranking data as in Tables 11, 12, or 13.
   (b) The advantages of grouping data into class intervals
2. Find the number such that: (use information in Table 11)
   (a) 25% of the scores are less than or equal to the number
   (b) half of the scores are less than or equal to the number
   (c) 75% of the scores are less than or equal to the number
3. Do the following for the set of test marks below:
   (a) Rank the marks according to numerical order.
   (b) Group the marks into intervals from 60 to 65; 65 to 70; etc. (the greater of the end-marks is included in each interval, e.g. 65 is included in the interval 60 to 65)
   (c) Make a frequency table showing the frequency for each interval.
   (d) Make a dot frequency diagram showing the frequency for each interval.
   Test Marks: 73, 67, 72, 88, 75, 89, 79, 81, 93, 76, 79, 82, 98, 90, 72, 70, 83, 78, 85, 84, 70, 73, 83, 78, 84, 92, 80, 69, 81, 78, 90, 93, 76, 78, 62, and 88.

5.14.5 The Frequency Histogram and the Cumulative Frequency Histogram

The frequency histogram is very similar to the dot frequency diagram. In place of the vertical columns of dots there are rectangles with width equal to the length of the group interval. The height of the rectangles is the frequency of the interval. Study the histograms below and compare them with the dot frequency diagrams of Table 13 which present the same data:

TABLE 14

FREQUENCY HISTOGRAMS
FREQUENCY OF SCORES IN FOUL-SHOOTING CONTEST

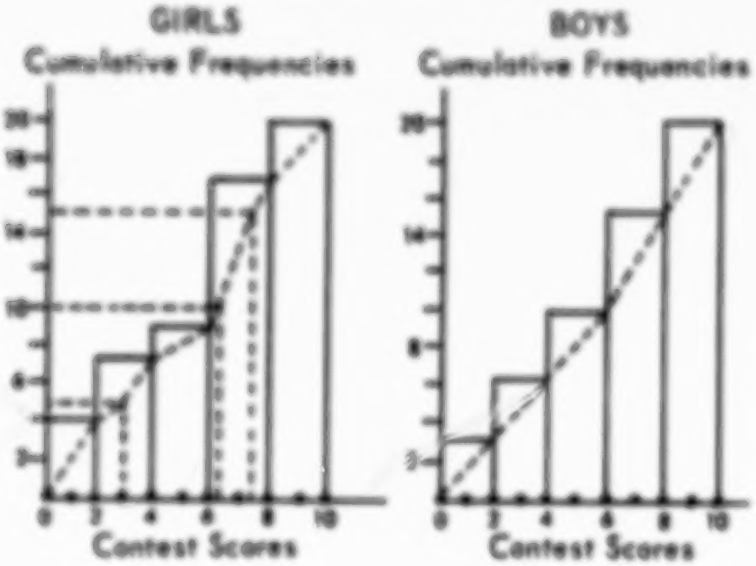


The cumulative frequency histogram is similar to the frequency histogram except that the second rectangle has height equal to the sum of the heights of the first two in the frequency histogram; the third is the sum of the first three etc. A table is included below which tabulates the cumulative frequencies to help interpret the cumulative frequency histograms:

TABLE 15 Cumulative Frequency Table for Foul-shooting Contest Scores

Class Interval	Frequency		Cumulative Frequency	
	Girls	Boys	Girls	Boys
0 - 2	4	3	4	3
2 - 4	3	3	7	6
4 - 6	2	4	9	10
6 - 8	8	5	17	15
8 - 10	3	5	20	20

TABLE 16 Cumulative Frequency Histograms for Foul-shooting Contest Scores



In both of the cumulative frequency histograms there is a dotted segment connecting the upper right corners of the rectangles. This set of segments is called a cumulative frequency polygon. It is helpful in determining the number below which 25% of the scores fall. This number is called the first quartile. It is likewise helpful in finding the comparable number for 50% of the scores or in fact any particular per cent of the scores.

Notice the horizontal dotted segments, going from 5, 10, and 15 on the vertical scale over to the polygon and then down to the horizontal scale. These determine the numbers which 25%, 50%, and 75% of the scores are less than or equal to. Other names for these numbers are first quartile, median, and third quartile respectively. They are very useful in classifying scores for comparison purposes.

5.14.6 Exercises

1. Use the set of test marks in 5.14.4 ex. 3(d):
   (a) Make a frequency histogram for the set of test marks.
   (b) Make a cumulative frequency histogram for the set of test marks.
   (c) Draw a cumulative frequency polygon in the graph of part (b).
   (d) Use the cumulative frequency polygon to find the first quartile, median and third quartile for the set of test marks.
2. Gather the following sets of data:
   (a) The heights to the nearest inch of each member of your class.
   (b) The ages to the nearest month of the members of your class.
   (c) The number of cars passing a certain point in some street during 20 five-minute intervals.
3. Present the data of Exercise 1 in the following ways:
   (a) A cumulative frequency table with a separate entry for each measure.
   (b) A cumulative frequency table with the data grouped into intervals.
   (c) A frequency histogram and polygon.
   (d) A cumulative frequency histogram and polygon.

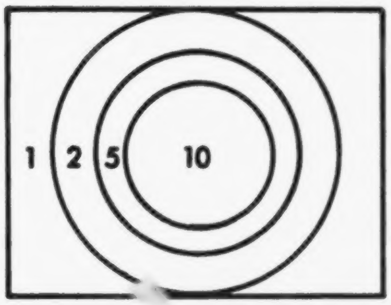
5.15 Summary

1. This chapter introduced several mathematical methods of predicting the outcome of activities in situations involving uncertainty. In the fish count problem, it is impractical to do more than estimate on the basis of incomplete knowledge. In die tossing, the outcome of a given trial can never be known in advance.
2. To assist in making good estimates or predictions, we performed a limited number of trials and observed the relative frequency of the various possible outcomes. We found that for a given experiment, the relative frequencies tended to stabilize as the number of trials increased.
3. On the basis of this stability of relative frequency, we made predictions of the likelihood or probability of events. The probability of an event—like the relative frequency—is a number assigned to the event. The number is
   A. 0, 1, or a number between 0 and 1;
   B. 0 for an impossible event;
   C. 1 for a certain event.

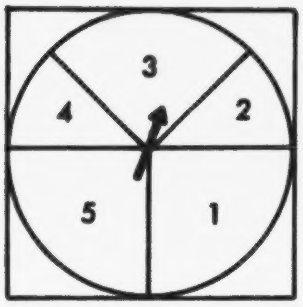
4. Furthermore, we found that probabilities and relative frequencies had the following properties:
   A. The sum of the probabilities (relative frequencies) of the outcomes in an outcome set is 1.
   B. If two events have no outcomes in common, the probability that one of the two will occur on a given trial is equal to the sum of the probabilities of the individual events.
   C. If an experiment has n equally probable outcomes and an event has s outcomes, the probability of the event is s/n.
5. The presentation of results is an important part of the experimentation done to determine relative frequencies. Frequencies and cumulative frequencies are reported in tables and a variety of kinds of graphs.
6. The instances in which "statistics" were misused pose an important problem for future study: What rules must be followed in correct analysis and reporting of situations involving uncertainty?

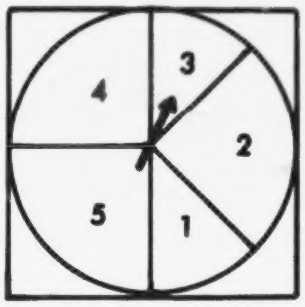
5.16 Review Exercises

1. List the members of an outcome set for each of the following experiments:
   - (a) Select two means of transportation from {bus, train, plane}.
   - (b) A dodecahedron (twelve-faced polyhedron) with faces numbered from 1 to 12 is rolled and the numeral on the up-face is observed.
   - (c) A pair of vowels is selected from the alphabet.
   - (d) Two darts are thrown at a target with four scoring possibilities:



   - (e) Three tags are selected from a box containing five blue tags and two red tags.
   - (f) Each of three people vote for Jones or Smith (but not both).
2. Two dials with sectors numbered from 1 to 5 are spun:





   - (a) Tabulate the outcome set.
   - (b) How many ordered pairs are in the outcome set?
   - (c) Assuming that each ordered pair in the outcome set is equally likely, what is $p(\{(2, 5)\})$?
   - (d) What is the probability that both dials will yield an even number?
   - (e) What is the probability that at least one of the dials will yield an even number?
   - (f) Make a rectangular arrangement of dots to represent the outcome set.
   - (g) Make up an outcome set based on the sums of the outcomes on each dial.
   - (h) Draw a line through the dots of the rectangular arrangement for which the sums of the outcomes are each six. Repeat for sums of five and seven. What is the probability for each of the above sums?
   - (i) Draw a line through the dots for which the differences of the outcomes are each three. Do the same for differences of one and zero. What is the probability for each of the above differences?
   - (j) Circle the dots for which at least one dial yields an even number.
3. Select two pages of a magazine article and separate the text into sets of ten lines.
   - (a) Find the relative frequency of the letter e, for each set of ten lines.
   - (b) Find the relative frequency of the letter x, for each set of ten lines.
   - (c) Compare the relative frequencies of e and x.
   - (d) Among the samples tested, were the relative frequencies for e fairly uniform? Answer the same question for x.
   - (e) What predictions could you make on the basis of the above investigation?
4. A coin and die are tossed simultaneously:
   - (a) Tabulate an outcome set which pairs each of the outcomes for the die with each for the coin.
   - (b) Assume that each simple outcome is equally likely.
   - (c) What is the probability that the die will show six?
   - (d) What is the probability that the die will not show six?
   - (e) What is the sum of the probabilities in (c) and (d)?
   - (f) For any event, E, what is the following sum, $P(E) + P(\text{not } E)$?
   - (g) What is the probability that the die will show six, given that the coin lands heads?
   - (h) Does the probability of the outcome six for the die depend on the event that the coin landed heads?
   - (i) What is the probability that the coin lands heads and the die shows six?
   - (j) Is the probability of the event described in (i) equal to the product of the probabilities for the coin landing heads and the die showing 6?
5. Describe two events, A and B, from the experiment in Exercise 4 that have no outcomes in common.
   - (a) What is $P(A)$? $P(B)$?
   - (b) What is $P(A) + P(B)$?
   - (c) What is the probability of the event, A occurs or B occurs?
   - (d) What generalization is illustrated by the answers to (a), (b) and (c)?
6. Make a table showing the number of children in the families of each student in your class. Then make a

table showing the relative frequency of one-child families, two-child families, etc. The illustrative table below shows that for a class of twenty students there were 5 one-child families so that the relative frequency for one-child families (in this sample) was 5/20 or 1/4

NUMBER OF CHILDREN per FAMILY IN
A SAMPLE OF TWENTY FAMILIES

Number of Children	Frequency	Relative Frequency
1	5	1/4
2	4	1/5
etc.	etc.	etc.

7. Make a similar chart for the distances from home to school for each student in your class, using the number of blocks as a measure. Group the data into class intervals. For example, all students living from 1 to 5 blocks from school might be grouped together, then 5 to 10, etc.

   Make a histogram and frequency polygon for this data.
8. Make a study of the time spent by each student in completing a particular homework assignment. Use five-minute intervals and decide on some good way to present the data. One way might be suggested by the following circular chart!